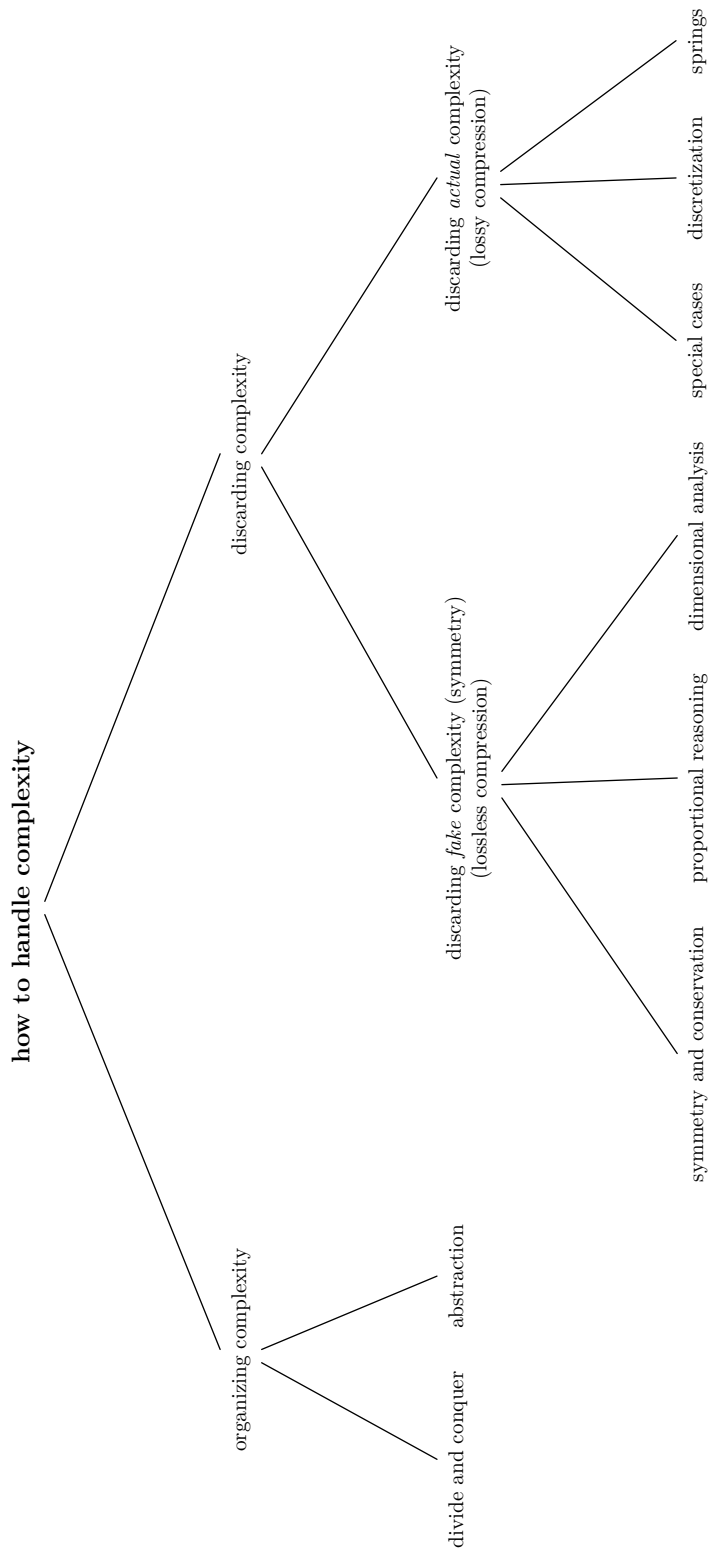


π	pi	3
G	Newton's constant	$7 \cdot 10^{-11} \text{ kg}^{-1} \text{ m}^3 \text{ s}^{-1}$
c	speed of light	$3 \cdot 10^8 \text{ m s}^{-1}$
k_B	Boltzmann's constant	$10^{-4} \text{ eV K}^{-1}$
e	electron charge	$1.6 \cdot 10^{-19} \text{ C}$
σ	Stefan-Boltzmann constant	$6 \cdot 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$
m_{sun}	Solar mass	$2 \cdot 10^{30} \text{ kg}$
R_{earth}	Earth radius	$6 \cdot 10^6 \text{ m}$
$\theta_{\text{moon/sun}}$	angular diameter	10^{-2}
ρ_{air}	air density	1 kg m^{-3}
ρ_{rock}	rock density	5 g cm^{-3}
$\hbar c$		200 eV nm
$L_{\text{vap}}^{\text{water}}$	heat of vaporization	2 MJ kg^{-1}
γ_{water}	surface tension of water	10^{-1} N m^{-1}
a_0	Bohr radius	$0.5 \text{ \AA}$
a	typical interatomic spacing	$3 \text{ \AA}$
N_A	Avogadro's number	$6 \cdot 10^{23}$
$\mathcal{E}_{\text{fat}}$	combustion energy density	9 kcal g^{-1}
E_{bond}	typical bond energy	4 eV
$\frac{e^2/4\pi\epsilon_0}{\hbar c}$	fine-structure constant α	10^{-2}
p_0	air pressure	10^5 Pa
ν_{air}	kinematic viscosity of air	$1.5 \cdot 10^{-5} \text{ m}^2 \text{ s}^{-1}$
ν_{water}	kinematic viscosity of water	$10^{-6} \text{ m}^2 \text{ s}^{-1}$
day		10^5 s
year		$\pi \cdot 10^7 \text{ s}$
F	solar constant	1.3 kW m^{-2}
AU	distance to sun	$1.5 \cdot 10^{11} \text{ m}$
P_{basal}	human basal metabolic rate	100 W
K_{air}	thermal conductivity of air	$2 \cdot 10^{-2} \text{ W m}^{-1} \text{ K}^{-1}$
K	... of non-metallic solids/liquids	$1 \text{ W m}^{-1} \text{ K}^{-1}$
K_{metal}	... of metals	$10^2 \text{ W m}^{-1} \text{ K}^{-1}$
c_p^{air}	specific heat of air	$1 \text{ J g}^{-1} \text{ K}^{-1}$
c_p	... of solids/liquids	$25 \text{ J mol}^{-1} \text{ K}^{-1}$



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Chapter 1

Preface

An approximate analysis is often more useful than an exact solution!

This counterintuitive thesis, the reason for this book, suggests two questions.

One question is: If science and engineering are about accuracy, how can approximate models be useful? They are useful because our minds are a small part of the world itself. When we represent a piece of the world in our minds, we discard many aspects – we make a model – in order that the model fit in our limited minds. An approximate model is all that we can understand. Making useful models means discarding less important information so that our minds may grasp the important features that remain.

This perhaps disappointing conclusion leads to a second question: Since every model is approximate, how do we choose useful approximations? The American psychologist William James said [15, p. 390]: ‘The art of being wise is the art of knowing what to overlook.’ This book therefore develops intelligence amplifiers: tools for discarding unimportant aspects of a problem and for selecting the important aspects.

These reasoning tools are of three types:

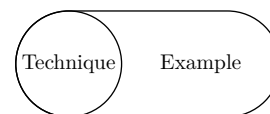
1. **Organizing complexity**
 - Divide and conquer
 - Abstraction
2. **Lossless compression**
 - Proportional reasoning
 - Conservation (box models)
 - Dimensionless groups
3. **Lossy compression**

- Special cases
- Spring models
- Discretization

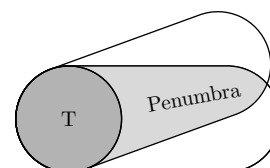
The first type of tool helps manage complexity. The second type helps remove complexity that is merely apparent. The third type helps discard complexity.

With these tools we explore the natural and manmade worlds, using examples from diverse fields such as quantum mechanics, general relativity, mechanical engineering, biophysics, recreational mathematics, and climate change. This diversity has two purposes. First, the diversity shows how a small toolbox can explain important features of the manmade and engineered worlds. The diversity provides a library of models for your own analyses.

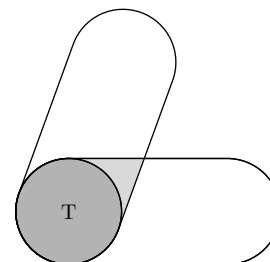
Second, the diversity separates the tool from the details of its use. A tool is difficult to appreciate abstractly, without an example. However, if you see only one use of a tool, the tool is difficult to distinguish from the example. An expert, familiar with the tool, knows where the idea ends and the details begin. But when you first learn a tool, you need to learn the bound-



ary. An answer is a second example. To the extent that the second example is similar to the first, the tool plus first use overlaps the tool plus second use. The overlap includes a penumbra around the tool. The penumbra is smaller than it is with only one example: Two uses delimit the boundaries of the tool more clearly than one example does.



More clarity comes using an example from a distant field. The penumbra shrinks, which separates the tool from examples of its use. For example, using dimensional analysis in a physics problem and an economics analysis clarifies what part of the illustration is specific to physics or economics and what part is transferable to other problems. Focus on the transferable ideas; they are useful in any career!



This book is designed for self study. Therefore, please try the problems. The problems are of two types. The first type are problems marked with a wedge in the margin. They are breathers during an analysis: a place to develop your understanding by working out the next steps in

6

an analysis. Those problems are answered in the subsequent text where you can check your thinking and my analysis – please let me know of any errors! The second type of problem, the numbered problems, give practice with the tools, extend a derivation, or develop a useful or enjoyable model. Most numbered problems have answers at the end of the book.

I hope that you find the tools, problems, and models useful in your career. And I hope that the diversity of examples connects with and aids your curiosity about how the world is put together.

Bon voyage!

Part 1

Organizing complexity

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The first solution to the messiness and complexity of the world, just as with the mess on our desks and in our living spaces, is to organize the complexity. Two techniques for organizing complexity are the subject of **Part 1**.

The first technique is divide-and-conquer reasoning: dividing a large problem into manageable subproblems. The second technique is abstraction: choosing compact representations that hide unimportant details in order to reveal important features. The next two chapters illustrate these techniques with many examples.

Chapter 2

Divide and conquer

How can ancient Sumerian history help us solve problems of our time?

From Sumerian times, and maybe before, every empire solved a hard problem – how to maintain dominion over resentful subjects. The obvious solution, brute force, costs too much: If you spend the riches of the empire just to retain it, why have an empire? But what if the resentful subjects would expend their energy fighting one another instead of uniting against their rulers? This strategy was summarized by Machiavelli [20, Book VI]:

A Captain ought. . . endeavor with every art to divide the forces of the enemy, either by making him suspicious of his men in whom he trusted, or *by giving him cause that he has to separate his forces*, and, because of this, become weaker. [my italics]

Or, in imperial application, divide the resentful subjects into tiny tribes, each too small to discomfort the empire. (For extra credit, reduce the discomfort by convincing the tribes to fight one another.)

Divide and conquer! As an everyday illustration of its importance, imagine taking all the files on your computer – mine claims to have 1901629 files – and stuffing all of them into one folder. Dividing those million odd files into a hierarchy is the only hope for conquering so much complexity.

This reasoning tool dissolves difficult problems into manageable pieces. It is a universal solvent for problems social, mathematical, engineering, and scientific.

To master a physical tool, we use and analyze it. We see what it can do, how it works, and maybe study the principles underlying its design. Similarly, the reasoning tool of divide and conquer, the subject of this chapter, is introduced using a mix of examples and theory. There are four examples: CDROM design, oil imports, bank robbery, and the UNIX operating system.

There are three sections of theory: how to increase confidence in estimates; how to represent divide-and-conquer reasoning graphically; and how to explain the uncanny accuracy of divide-and-conquer reasoning.

2.1 Example 1: CDROM design

Our first use of the divide-and-conquer tool is from electrical engineering and information theory:

► *How far apart are the pits on a compact disc (CD) or CDROM?*

Divide the finding the spacing into two subproblems: (1) estimating the CD's area and (2) estimating its data capacity. The area is roughly $(10\text{ cm})^2$ because each side is roughly 10 cm long. The actual length, according to a nearby ruler, is 12 cm; so 10 cm is an underestimate. However, (1) the hole in the center reduces the disc's effective area; and (2) the disc is circular rather than square. So $(10\text{ cm})^2$ is a reasonable and simple estimate of the disc's pitted area.

The data capacity, according to a nearby box of CDROM's, is 700 megabytes (MB). Each byte is 8 bits, so here is the capacity in bits:

$$700 \cdot 10^6 \text{ bytes} \times \frac{8 \text{ bits}}{1 \text{ byte}} \sim 5 \cdot 10^9 \text{ bits.}$$

Each bit is stored in one pit, so their spacing is a result of arranging them into a lattice that covers the $(10\text{ cm})^2$ area. 10^{10} pits would need 10^5 rows and 10^5 columns, so the spacing between pits is roughly

$$d \sim \frac{10\text{ cm}}{10^5} \sim 1\text{ }\mu\text{m.}$$

That calculation was simplified by rounding up the number of bits from $5 \cdot 10^9$ to 10^{10} . The factor of 2 increase means that $1\text{ }\mu\text{m}$ underestimates the spacing by a factor of $\sqrt{2}$, which is roughly 1.4: The estimated spacing is $1.4\text{ }\mu\text{m}$.

Finding the capacity on a box of CDROM's was a stroke of luck. But fortune favors the prepared mind. To prepare the mind, here is a divide-and-conquer estimate for the capacity of a CDROM – or of an audio CD, because data and audio discs differ only in how we interpret the information. An audio CD's capacity can be estimated from three quantities: the playing time, the sampling rate, and the sample size (number of bits per sample).

► *Estimate the playing time, sampling rate, and sample size.*

Here are estimates for the three quantities:

1. *Playing time.* A typical CD holds about 20 popular-music songs each lasting 3 minutes, so it plays for about 1 hour. Confirming this estimate is the following piece of history. Legend, or urban legend, says that the designers of the CD ensured that it could record Beethoven's Ninth Symphony. At most tempos, the symphony lasts 70 minutes.
2. *Sampling rate.* I remember the rate: 44 kHz. This number can be made plausible using information theory and acoustics.

First, acoustics. Our ears can hear frequencies up to 20 kHz (slightly higher in youth, slightly lower in old age). To reproduce audible sounds with high fidelity, the audio CD is designed to store frequencies up to 20 kHz: Why ensure that Beethoven's Ninth Symphony can be recorded if, by skimping on the high frequencies, it sounds like was played through a telephone line?

Second, information theory. Its fundamental theorem, the Nyquist–Shannon sampling theorem, says that reconstructing a 20 kHz signal requires sampling at 40 kHz – or higher. High rates simplify the anti-alias filter, an essential part of the CD recording system. However, even an 80 kHz sampling rate exceeded the speed of inexpensive electronics when the CD was designed. As a compromise, the sampling-rate margin was set at 4 kHz, giving a sampling rate of 44 kHz.

3. *Sample size.* Each sample requires 32 bits: two channels (stereo) each needing 16 bits per sample. Sixteen bits per sample is a compromise between the utopia of exact volume encoding (infinity bits per sample per channel) and the utopia of minimal storage (1 bit per sample per channel). Why compromise at 16 bits rather than, say, 50 bits? Because those bits would be wasted unless the analog components were accurate to 1 part in 2^{50} . Whereas using 16 bits requires an accuracy of only 1 part in 2^{16} (roughly 10^5) – attainable with reasonably priced electronics.

The preceding three estimates – for playing time, sampling rate, and sample size – combine to give the following estimate:

$$\text{capacity} \sim 1 \text{ hr} \times \frac{3600 \text{ s}}{1 \text{ hr}} \times \frac{4.4 \times 10^4 \text{ samples}}{1 \text{ s}} \times \frac{32 \text{ bits}}{1 \text{ sample}}.$$

This calculation is an example of a conversion. The starting point is the 1 hr playing time. It is converted into the number of bits stepwise. Each step is a multiplication by unity – in a convenient form. For example, the first form of unity is 3600 s/1 hr; in other words, 3600 s = 1 hr. This equivalence is

a truth generally acknowledged. Whereas a particular truth is the second factor of unity, $4.4 \cdot 10^4$ samples/1 s, because the equivalence between 1 s and $4.4 \cdot 10^4$ samples is particular to this example.

Problem 2.1 General or particular?

In the conversion from playing time to bits, is the third factor a general or particular form of unity?

Problem 2.2 US energy usage

In 2005, the US economy used 100 quads. One quad is one quadrillion (10^{15}) British thermal units (BTU's); one BTU is the amount of energy required to raise the temperature of one pound of liquid water by one degree Fahrenheit. Using that information, stepwise convert the US energy usage into familiar units such as kilowatt-hours.

What is the corresponding power consumption (in Watts)?

To evaluate the capacity product in your head, divide it into two subproblems – the power of ten and everything else:

1. *Powers of ten.* They are, in most estimates, the big contributor; so, I always handle powers of ten first. There are eight of them: The factor of 3600 contributes three powers of ten; the 4.4×10^4 contributes four; and the 2×16 contributes one.
2. *Everything else.* What remains are the mantissas – the numbers in front of the power of ten. These moderately sized numbers contribute the product $3.6 \times 4.4 \times 3.2$. The mental multiplication is eased by collapsing mantissas into two numbers: 1 and 'few'. This number system is designed so that 'few' is halfway between 1 and 10; therefore, the only interesting multiplication fact is that $(\text{few})^2 = 10$. In other words, 'few' is approximately 3. In $3.6 \times 4.4 \times 3.2$, each factor is roughly a 'few', so $3.6 \times 4.4 \times 3.2$ is approximately $(\text{few})^3$, which is 30: one power of 10 and one 'few'. However, this value is an underestimate because each factor in the product is slightly larger than 3. So instead of 30, I guess 50 (the true answer is 50.688). The mantissa's contribution of 50 combines with the eight powers of ten to give a capacity of $5 \cdot 10^9$ bits – in surprising agreement with the capacity figure on a box of CDROM's.

► Find the examples of divide-and-conquer reasoning in this section.

Divide-and-conquer reasoning appeared three times in this section:

1. spacing dissolved into capacity and area;
2. capacity dissolved into playing time, sampling rate, and sample size; and
3. numbers dissolved into mantissas and powers of ten.

These uses illustrate important maneuvers using the divide-and-conquer tool. Further practice with the tool comes in subsequent sections and in the problems. However, we have already used the tool enough to consider how to use it with finesse. So, the next two sections are theoretical, in a practical way.

2.2 Theory 1: Multiple estimates

After estimating the pit spacing, it is natural to wonder: How much can we trust the estimate? Did we make an embarrassingly large mistake? Making reliable estimates is the subject of this section.

In a familiar instance of searching for reliability, when we mentally add a list of numbers we often add the numbers first from top to bottom. For example: *12 plus 15 is 27; 27 plus 18 is 45*. Then, to check the result, we add the numbers in reverse: *18 plus 15 is 33; 33 plus 12 is 45*. When the two totals agree, as they do here, each is probably correct: The chance is low that both additions contain an error of exactly the same amount.

Redundancy, it seems, reduces errors. Mindless redundancy, however, offers little protection. As an example, if we repeatedly add the numbers from top to bottom, we are likely to repeat our mistakes from the first attempt. Similarly, reading your rough drafts several times usually means repeatedly overlooking the same spelling, grammar, or logic faults. Instead, put the draft in a drawer for a week, then look at it, or ask a colleague or friend – in both cases, use fresh eyes.

This robustness heuristic was in the Laser Interferometric Gravitational Observatory (LIGO), an extremely sensitive system to detect gravitational waves. It contains one detector in Washington and a second in Louisiana. The LIGO fact sheet explains the redundancy:

Local phenomena such as micro-earthquakes, acoustic noise, and laser fluctuations can cause a disturbance at one site, simulating a gravitational wave event, but such disturbances are unlikely to happen simultaneously at widely separated sites.

Robustness, in short, comes from *intelligent* redundancy.

This principle helps us make reliable, robust estimates. Not only should we use several methods, we should make the methods different from one another; for example, make the methods use unrelated knowledge and information. This approach is another use of divide and conquer (which may explain why the approach belongs in this chapter): The hard problem of making a robust estimate becomes several simpler subproblems – one per estimation method.

So, to supplement the divide-and-conquer estimate for the pit spacing ([Section 2.1](#)), here are two intelligently redundant methods:

1. An optics method is based on turning over a CD to enjoy and explain the brilliant, shimmering colors. The colors are caused by how the pits diffract different wavelengths of light. (Diffraction is beautifully explained in Feynman's *QED* [13].) For a pristine example of diffraction, find a red-light laser pointer, the kind often used for presentations. When you shine it onto the back of a CD, you'll see several red dots on the wall. These dots are separated by the diffraction angle. This angle, we learn from optics, depends on the wavelength (or color): It is λ/D , where λ is the wavelength and D is the pit spacing. Since light contains a spectrum of colors, each color diffracts by its own angle. Tilting the disc changes the mix of spots – of colors – that reach your eye, creating the shimmering colors.

Their brilliance hints that the diffraction angles are significant – meaning that they are comparable to 1 rad. To estimate the angle more precisely, and lacking a laser pointer, I took a CD to a sunny spot and noted what appeared on the nearest wall: There was a sunny circle, the reflected image of the CD, surrounded by a diffracted rainbow. Relative to the reflected image, the rainbow appeared at an angle of roughly 30° or 0.5 rad. This data along with the diffraction relation $\theta \sim \lambda/D$ implies that the pit spacing is approximately 2λ . Since visible-light wavelengths range from $0.35 \mu\text{m}$ to $0.7 \mu\text{m}$ – let's call it $0.5 \mu\text{m}$ – I estimate the pit spacing to be $1 \mu\text{m}$.

2. A hardware method is based on how a CD player or a CDRom drive reads data. It scans the disc with a tiny laser that emits – I seem to remember – near-infrared radiation. The *infrared* means that the radiation's wavelength is longer than the wavelength of red light; the *near* indicates that its wavelength is close to the wavelength of red light. Therefore, *near infrared* means that the wavelength is only slightly longer than the wavelength of red light. For the laser to read the pits, its wavelength should be smaller than the pit spacing or size. Since red light has

a wavelength of roughly 700 nm, I'll guess that the laser has a wavelength of 800 nm or 1000 nm and that the pit spacing is slightly larger – 1 μm . (The actual wavelength is 780 nm.)

Three significantly different methods give comparable estimates: 1.4 μm (capacity), 1 μm (optics), and 1 μm (hardware). Therefore, we have probably not committed a blunder in any method. To make that argument concrete, imagine that the true spacing is 0.1 μm . Then three *independent* methods all contain an error of a factor of 10 – and each time producing an overestimate. Such a coincidence is not common. Although any method can contain errors – the world is infinite but our abilities are finite – the errors would not often agree in sign (being an over- or underestimate) and magnitude.

The lesson – that intelligent redundancy produces robustness – seems plausible now, I hope. But the proof of the pudding is in the eating: What is the true pit spacing? It depends whether you mean the radial or the transverse spacing. The data pits lie on a tremendously long spiral track whose 'rings' lie 1.6 μm apart. Along the track, the pits lie 0.9 μm apart. So, the spacing is between 0.9 and 1.6 μm ; if you want just one value, let it be the midpoint, 1.3 μm . We made a tasty pudding!

Problem 2.3 Robust addition

The text offered addition as an example of intelligent redundancy: We often verify an addition by redoing the sum from bottom to top. Analyze this practice using simple probability models. Is it indeed an example of intelligent redundancy?

Problem 2.4 Intelligent redundancy

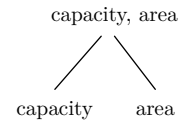
Think of and describe a few real-life examples of intelligent redundancy.

2.3 Theory 2: Tree representations

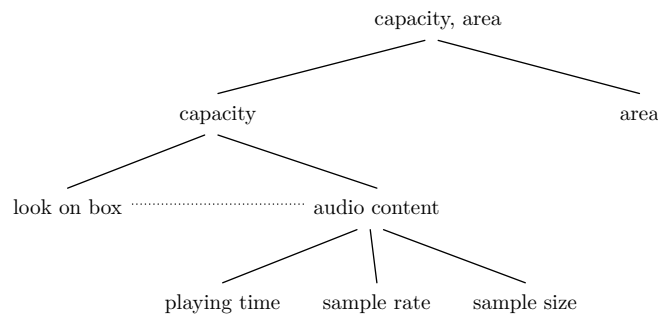
Tasty though the estimation pudding may be, its recipe is long and detailed. It is hard to follow – even for its author. Although I wrote the analysis, I cannot quickly recall all its pieces; rather, I must remind myself of the pieces by looking over the text. As I do, I am reminded that sentences, paragraphs, and pages do not compactly represent a divide-and-conquer estimate.

Linear, sequential information does not match the estimate's structure. Its structure is hierarchical – with answers constructed from solving smaller problems, which might be constructed from even solving still smaller problems – and its most compact representation is as a tree.

As an example, let's construct the tree representing the elaborate divide-and-conquer estimate for a CDROM's pit spacing ([Section 2.1](#)). The tree's root is 'capacity, area', a two-word tag reminding us of the method underlying the estimate. The estimate dissolves into finding two quantities – the capacity and area – so the tree's root sprouts two branches.



Of the two new leaves, the area is easy to estimate without explicitly subdividing into smaller problems, so the 'area' node remains a leaf. To estimate the capacity – rather, to estimate the capacity reliably – we used intelligent redundancy: (1) looking on a CDROM box; and (2) estimating how many bits are required to represent the music that fits on an audio CD. The second method subdivided into three estimates: for the playing time, sample rate, and sample size. Accordingly, the 'capacity' node sprouts new branches – and a new connector:



The dotted horizontal line indicates that its endpoints redundantly evaluate their common parent (see [Section 2.2](#)). Just as a crossbar strengthens a structure, the crossing line indicates the extra reliability of an estimate based on redundant methods.

The next step in representing the estimate is to include estimates at the five leaves:

1. capacity on a box of CDROM's: 700 MB;
2. playing time: roughly one hour;
3. sampling rate: 44 kHz;

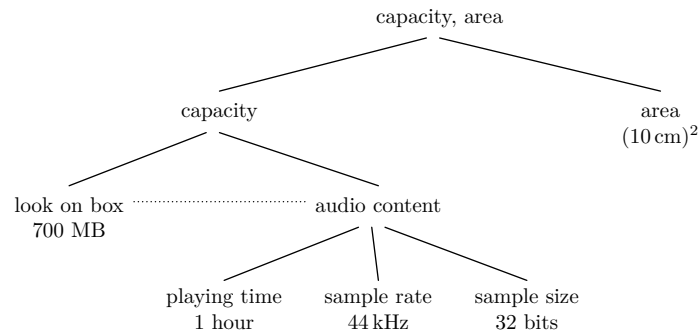
16

2.3. Theory 2: Tree representations

4. sample size: 32 bits;

5. area: $(10\text{ cm})^2$.

Here is the quantified tree:



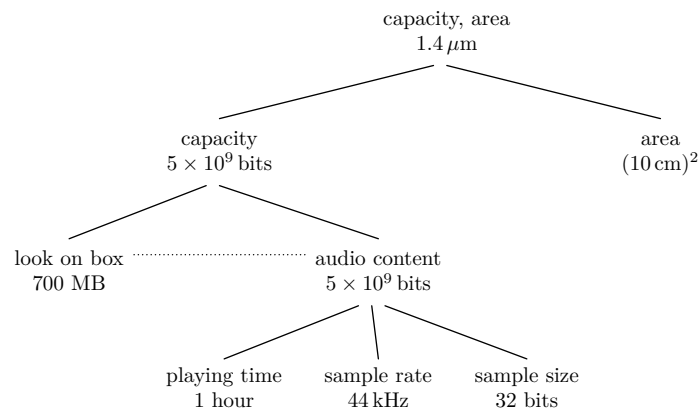
The final step is to propagate estimates upward, from children to parent, until reaching the root.

► Draw the resulting tree.

Here are estimates for the nonleaf nodes:

1. *audio content*. It is the product of playing time, sample rate, and sample size: $5 \cdot 10^9$ bits.
2. *capacity*. The look-on-box and audio-content methods agree on the capacity: $5 \cdot 10^9$ bits.
3. *pit spacing computed from capacity and area*. At last, the root node! The pit spacing is $\sqrt{A/N}$, where A is the area and N is the capacity. The spacing, using that formula, is roughly $1.4\text{ }\mu$.

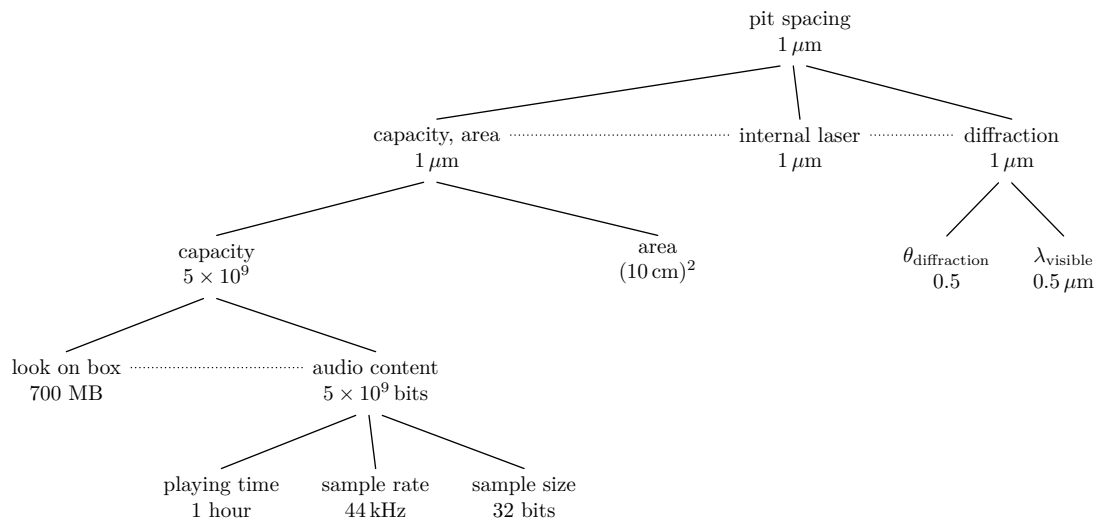
Propagating estimates from leaf to root gives the following tree:



This tree is far more compact than the sentences, equations, and paragraphs of the original analysis in [Section 2.1](#). The comparison becomes even stronger by including the alternative estimation methods in [Section 2.2](#): (1) the wavelength of the internal laser, and (2) diffraction to explain the shimmering colors of a CD.

► Draw a tree that includes these methods.

The wavelength method depends on just quantity, the wavelength of the laser, so its tree has just that one node. The diffraction method depends on two quantities, the diffraction angle and the wavelength of visible light, so its tree has those two nodes as children. All three trees combine into a larger tree that represents the entire analysis:



This tree summarizes the whole analysis of [Section 2.1](#) and [Section 2.2](#) – in one figure. The compact representation make it possible to grasp the analysis in one glance. It makes the whole analysis easier to understand, evaluate, and perhaps improve.

2.4 Example 2: Oil imports

For practice, here is a divide-and-conquer estimate using trees throughout:

► How much oil does the United States import (in barrels per year)?

One method is to subdivide the problem into three quantities:

- estimate how much oil is used every year by cars;
- increase the estimate to account for non-automotive uses; and
- decrease the estimate to account for oil produced in the United States.

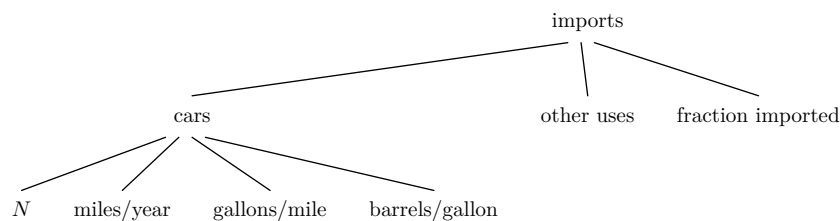
Here is the corresponding tree:

The first quantity requires the longest analysis, so begin with the second and third quantities. Other than for cars, oil is used for other modes of transport (trucks, trains, and planes); for heating and cooling; and for manufacturing hydrocarbon-rich products (fertilizer, plastics, pesticides). To guess the fraction of oil used by cars, there are two opposing tendencies: (1) the idea that the non-automotive uses are so important, pushing the fraction toward zero; (2) the idea that the automotive uses are so important, pushing the fraction toward unity.

Both ideas seem equally plausible to me; therefore, I guess that the fraction is roughly one-half; and, to account for non-automotive uses, I will double the estimate of oil consumed by cars.

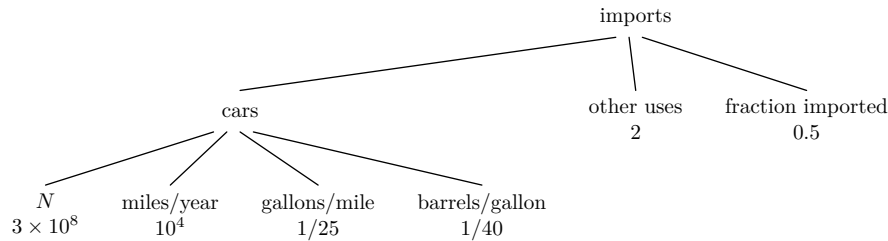
Imports are a large fraction of total consumption, otherwise we would not read so much in the popular press about oil production in other countries, and about our growing dependence on imported oil. Perhaps one-half of the oil usage is imported oil. So I need to halve the total use to find the imports.

The third leaf, cars, is too complex to guess a number immediately. So divide and conquer. One subdivision is into number of cars, miles driven by each car, miles per gallon, and gallons per barrel:



Now guess values for the unnumbered leaves. There are 3×10^8 people in the United States, and it seems as if even babies own cars. As a guess, then, the number of cars is $N \sim 3 \times 10^8$. The annual miles per car is maybe 15,000. But the N is maybe a bit large, so let's lower the annual miles estimate to 10,000, which has the additional merit of being easier to handle. A typical mileage would be 25 miles per gallon. Then comes the tricky part: How large is a barrel? One method to estimate it is that a barrel costs about \$100,

and a gallon of gasoline costs about \$2.50, so a barrel is roughly 40 gallons. The tree with numbers is:



All the leaves have values, so I can propagate upward to the root. The main operation is multiplication. For the 'cars' node:

$$3 \times 10^8 \text{ cars} \times \frac{10^4 \text{ miles}}{1 \text{ car-year}} \times \frac{1 \text{ gallon}}{25 \text{ miles}} \times \frac{1 \text{ barrel}}{40 \text{ gallons}} \sim 3 \times 10^9 \text{ barrels/year.}$$

The two adjustment leaves contribute a factor of $2 \times 0.5 = 1$, so the import estimate is

$$3 \times 10^9 \text{ barrels/year.}$$

For 2006, the true value (from the US Dept of Energy) is 3.7×10^9 barrels/year – only 25 higher than the estimate!

2.5 Theory 3: Estimating accuracy

How does divide-and-conquer reasoning produce such accurate estimates? Alas, this problem is hard to analyze directly because we do not know accuracy in advance. But we can analyze a related problem: how divide-and-conquer reasoning increases our confidence in an estimate or, more precisely, decreases our uncertainty.

The answer is that it works by subdividing a quantity about which we know little into several quantities about which we know more. Even if we need many subdivisions before we reach reliable information, the increased certainty outweighs the small penalty for combining many quantities.

To explain that telegraphic answer, I will analyze a short estimation problem using divide-and-conquer done in slow motion, then apply the lessons to the oil-imports estimate.

The slow-motion problem is to estimate area of a sheet of A4 paper. On first thought, even looking at a sheet I have no clue about its area! On second thought, I know something. For example, the area is certainly more than

1 cm^2 and less than 10^5 cm^2 . That wide range makes it hard to be wrong, but it is also too wide to be useful. To narrow the range, I drew a small square with an area of roughly 1 cm^2 and guessed how many squares fit on the sheet: probably at least a few hundred and probably at most a few thousand. Turning ‘few’ into 3, I offer 300 cm^2 to 3000 cm^2 as a plausible range for the area.

Now compare that range to the range after doing divide and conquer. So, subdivide the area into the width and height: two quantities about which my knowledge is more precise than it is about area itself. The extra precision has a general reason and a reason specific to this problem. The general reason is that we have more experience with lengths than areas: Which is the more familiar quantity, your height or your cross-sectional area? So our length estimates are usually more accurate than our area estimates.

The reason specific to this problem is that A4 paper is the European equivalent of standard American paper, known to computers and laser printers as ‘letter’ paper and known commonly in the United States as ‘eight-and-a-half by eleven’ (inches!). In metric units, the dimensions are $21.59 \text{ cm} \times 27.94 \text{ cm}$. If A4 paper were identical to letter paper, I could now compute its exact area. However, A4 paper is, I remember from living in England, slightly thinner and longer than letter paper. I forget the exact differences between the dimensions of A4 and letter paper, hence the remaining uncertainty: I’ll guess that the width lies in the range $19 \dots 21 \text{ cm}$ and the length lies in the range $28 \dots 32 \text{ cm}$.

The next problem is to combine the plausible ranges for the height and width into the plausible range for the area. A first guess, because the area is the product of the width and height, is to multiply the endpoints of the width and height ranges:

$$A_{\min} = 19 \text{ cm} \times 28 \text{ cm} = 532 \text{ cm}^2;$$

$$A_{\max} = 21 \text{ cm} \times 32 \text{ cm} = 672 \text{ cm}^2.$$

This method turns out to overextend the range – a mistake that I correct later – but even the overextended range spans only a factor of 1.26 whereas the starting range of $300 \dots 3000 \text{ cm}^2$ spans a factor of 10. Divide and conquer significantly narrowed the range by replacing quantities about which we have little knowledge, such as the area, with quantities about which we have more knowledge.

The second bonus, which I now quantify correctly, is that subdividing into many quantities carries only a small penalty, smaller than suggested by naively multiplying endpoints. The naive method overestimates the range

because it assumes the worst. To see how, imagine an extreme case: estimating a quantity that is the product of ten factors, each that you know to within a factor of 2 (in other words, each plausible range is a factor of 4). Is your plausible range for the final quantity a factor of $4^{10} \approx 10^6$?! That conclusion is terribly pessimistic. A more likely case is that a few of the ten estimates will be too large and a few too small, and therefore that several errors will cancel.

To quantify and fix this pessimism, I will explain plausible ranges using probabilities. Probabilities are the tool for this purpose. They reflect incomplete knowledge *not* frequencies in a random experiment; see [16] for a book-length discussion and application of this fundamental point.

To illustrate the probabilistic description, start with the proposition

$H \equiv$ The area of A4 lies in the range $300 \dots 3000 \text{ cm}^2$.

and information

$I \equiv$ What I know about the area *before* using divide and conquer.

Now I want to know the conditional probability $P(H|I)$: the probability of H given my knowledge before trying divide and conquer. There is no algorithm known for computing this probability in such a complicated problem situation. How, for example, do we represent my state of knowledge? The best we can do in these cases is to introspect or, in plain English, to talk to our gut.

My gut is the organ with the most access to my knowledge and its incompleteness, and it tells me that I would feel mild surprise but not shock if I learned that the true area lay outside the range $300 \dots 3000 \text{ cm}^2$. The surprise suggests that $P(H|I)$ is larger than $1/2$. The mildness of the surprise suggests that $P(H|I)$ is not much larger than $1/2$. I'll quantify it as $P(H|I) = 2/3$: I would give 2-to-1 odds that the true area is within the plausible range. Throughout this book I'll use a rough 2-to-1 odds range to quantify a plausible range. I could have used a 1-to-1 odds range instead, but the 2-to-1 odds range will help give plausible ranges an intuitive interpretation as a region on a log-normal distribution. That interpretation will help quantify how to combine plausible ranges.

For the moment, I need only the idea that the plausible range contains roughly $2/3$ of the probability. With a further assumption of symmetry, the plausible range $300 \dots 3000 \text{ cm}^2$ represents the following probabilities:

$$\begin{aligned} P(A < 300 \text{ cm}^2) &= 1/6; \\ P(300 \text{ cm} \leq A \leq 3000 \text{ cm}^2) &= 2/3; \\ P(A > 3000 \text{ cm}^2) &= 1/6. \end{aligned}$$

Here is the corresponding picture with width proportional to probability:

	$p \approx 1/6$	$p \approx 2/3$	$p \approx 1/6$
A	$< 300 \text{ cm}^2$	$300 \dots 3000 \text{ cm}^2$	$> 3000 \text{ cm}^2$

For the height h and width w , after doing divide and conquer and using the similarity between A4 and letter paper, the plausible ranges are $28 \dots 32 \text{ cm}$ and $19 \dots 21 \text{ cm}$ respectively. Here are their probability interpretations:

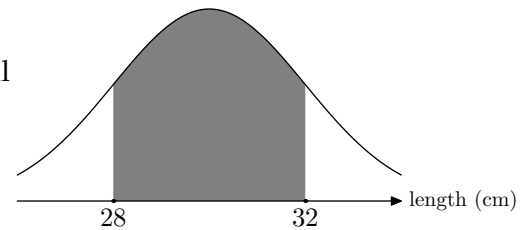
	$p \approx 1/6$	$p \approx 2/3$	$p \approx 1/6$
h	$< 28 \text{ cm}$	$28 \dots 32 \text{ cm}$	$> 32 \text{ cm}$

	$p \approx 1/6$	$p \approx 2/3$	$p \approx 1/6$
w	$< 19 \text{ cm}$	$19 \dots 21 \text{ cm}$	$> 21 \text{ cm}$

Computing the plausible range for the area requires a complete probabilistic description of a plausible range. There is a correct answer to this question – at least if, like me, you are an objective Bayesian – and it depends on the information available to the person giving the range. But no one knows the exact recipe to deduce probabilities from the complex, diffuse, seemingly contradictory information lodged in a human mind.

The best that we can do for now is to guess a reasonable and convenient probability distribution. I will use a log-normal distribution meaning that the uncertainty in the quantity's logarithm has a normal (or Gaussian) distribution. As an example, the figure shows the probability distribution for the length of A4 length

(after taking into account the similarity to letter paper). The shaded range is the the so-called one-sigma range $\mu - \sigma$ to $\mu + \sigma$. It contains 68% of the probability – a figure conveniently close to $2/3$. So to convert a plausible range to a log-normal distribution, use the lower and upper endpoints of the plausible range as $\mu - \sigma$ to $\mu + \sigma$. The peak of the distribution – the



most likely value – occurs midway between the endpoints. Since ‘midway’ is on a logarithmic scale, the midpoint is at $\sqrt{28 \times 32}$ cm or approximately 29.93 cm.

Problem 2.5 Midpoints

The midpoint on the log scale is also known as the geometric mean. Show that it is never greater than the midpoint on the usual scale (which is also known as the arithmetic mean). Can the two midpoints ever be equal?

The log-normal distribution supplies the missing information required to combine plausible ranges. When adding independent quantities, you add their means and their variances. So when multiplying independent quantities, add the means and variances in the logarithmic space. Here is the resulting recipe. Let the plausible range for h be $l_1 \dots u_1$ and the plausible range for w be $l_2 \dots u_2$. First compute the geometric mean (midpoint) of each range:

$$\begin{aligned}\mu_1 &= \sqrt{l_1 u_1}; \\ \mu_2 &= \sqrt{l_2 u_2}.\end{aligned}$$

The midpoint of the range for $A = hw$ is the product of the two midpoints: $\mu = \mu_1 \mu_2$.

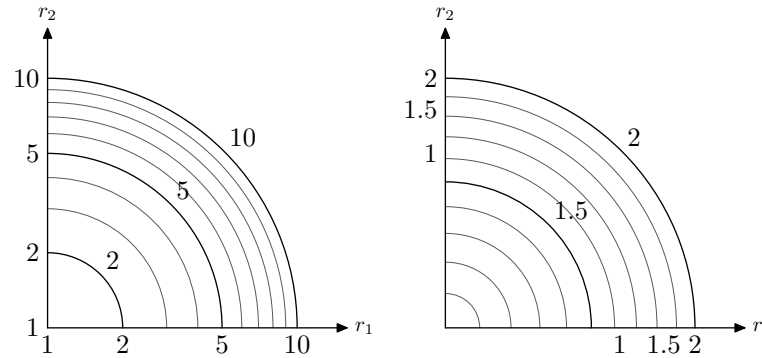
To compute the plausible range, first compute the ratios measuring the width of the ranges:

$$\begin{aligned}r_1 &= u_1/l_1; \\ r_2 &= u_2/l_2.\end{aligned}$$

These ratios measure the width of the ranges. The combined ratio – that is, the ratio of endpoints for the combined plausible range – is

$$r = \exp \left(\sqrt{(\ln r_1)^2 + (\ln r_2)^2} \right).$$

For approximate range calculations, the following contour graphs often provide enough accuracy:



After finding the range, choose the lower and upper endpoints l and u to make $u/l = r$ and $\sqrt{lu} = \mu$. In other words, the plausible range is

$$\frac{\mu}{\sqrt{r}} \dots \mu\sqrt{r}.$$

- Use a simple example to check that this method produces reasonable results.
- Apply the method to find the plausible range for the area of an A4 sheet.

Problem 2.6 Deriving the ratio

Use Bayes theorem to confirm this method for combining plausible ranges.

Let's check this method in a simple example: width and height ranges of $1 \dots 2$ m. What is the plausible range for the area? The naive approach of multiplying endpoints produces a plausible range of $1 \dots 4$ m² – a span of a factor of 4. The correct range should be narrower. Indeed, assuming the log-normal distribution, the range spans a factor of

$$\exp\left(\sqrt{2 \times (\ln 2)^2}\right) \approx 2.67.$$

This span and the midpoint determine the range. The area midpoint is the product of the width and height midpoints, each of which is $\sqrt{2}$ m. So the midpoint is 2 m². The correct endpoints are therefore

$$\frac{2 \text{ m}^2}{\sqrt{2.67}} \dots 2 \text{ m}^2 \times \sqrt{2.67}$$

or $1.23 \dots 3.27$ m². In other words, I assign roughly a $1/6$ probability that the area is less than 1.23 m² and roughly a $1/6$ probability that it is greater

than 3.27 m^2 . Those conclusions seem reasonable when using such uncertain knowledge of length and width.

Having checked that the method is reasonable, it is time to test it in the original illustrative problem: the plausible area range for an A4 sheet. The naive plausible range was $532 \dots 672 \text{ cm}^2$, and the correct plausible range will be narrower. Indeed, the log-normal method gives the narrower area range of $550 \dots 650 \text{ cm}^2$ with a best guess (most likely value) of 598 cm^2 . How did we do? The true area is exactly 2^{-4} m^2 or 625 cm^2 because – I remembered only after doing this calculation! – An paper is constructed to have one-half the area of A(n – 1) paper, with A0 paper having an area of 1 m^2 . The true area is only 5% larger than the best guess, suggesting that we used accurate information about the length and width; and it falls within the plausible range but not right at the center, suggesting that the method for computing the plausible range is neither too daring nor too conservative.

Problem 2.7 Volume of a room

Estimate the volume of your favorite room, giving your plausible range before and after using divide and conquer.

The analysis of combining ranges illustrates the two crucial points about divide-and-conquer reasoning. First, the main benefit comes from subdividing vague knowledge (such as the area itself) into pieces about which our knowledge is accurate. Second, this benefit swamps the small accuracy penalty from combining many quantities into one.

To confirm these lessons, examine the benefit of divide-and-conquer reasoning in the example from [Section 2.4](#): estimating the annual US oil imports. To quantify the benefit, I compare my plausible ranges before and after using divide and conquer.

Before I use divide and conquer, I have almost no idea what the oil imports are, and I am scared even to guess. To nudge me along, I imagine a mugger demanding, ‘Your guess or your life!’ In which case I counteroffer with, ‘Can I give you a range instead of a number? I’d be surprised if the annual imports are less than 10^7 barrels/yr or more than 10^{12} barrels/yr.’ The imaginary mugger, being my own creation, always accepts my offer.

Problem 2.8 Your range

What is your plausible range for the annual oil imports?

I need little prodding to narrow my plausible range using divide-and-conquer reasoning. It required making several estimates:

1. N_{people} : US population;
 2. f_{car} : cars per person;
 3. l : average distance that a car is driven
 4. m : average gas mileage;
 5. V : volume of a barrel;
 6. f_{other} : factor to multiply auto consumption to include all other consumption;
 7. f_{imported} : fraction of oil that is imported.
-

Problem 2.9 Your ranges

Give your plausible range for each quantity, i.e. the range for which you assign a two-thirds probability that the true value lies within the range.

Here are my plausible ranges with a few notes of explanation:

1. N_{people} : 290–310 million. I recently read in the newspaper that the US population just reached the milestone of 300 million. How much should I believe what I read in the paper? The media lie when it serves the powerful, but I cannot find any reason to lie about the US population, so I trust the figure, and throw in a bit of uncertainty to reflect the difficulties encountered in counting the population (e.g. what about undocumented immigrants, who are unlikely to fill out census forms?).
2. f_{car} : 0.5–1.5.
3. l : $7 \cdot 10^3$ – $20 \cdot 10^3$ mi. Some books assessing used cars consider a low-mileage car to have less than 10^4 mi per year of age. So I guess that the average is somewhat larger than 10^4 mi/yr. But I am not confident of my recollection or the deduction, so my plausible range spans a factor of 3.
4. m : 15–40 miles/gallon;

5. V : 30–60 gallons;
6. f_{other} : 1.5–3;
7. f_{imported} : 0.3–0.8.

► *What is the resulting plausible range for the oil imports?*

Now combine the ranges using the method we used for the area of a sheet of A4 paper. That method produces the following plausible range:

$$1.0 \dots 3.1 \dots 9.6 \cdot 10^9 \text{ barrels/year.}$$

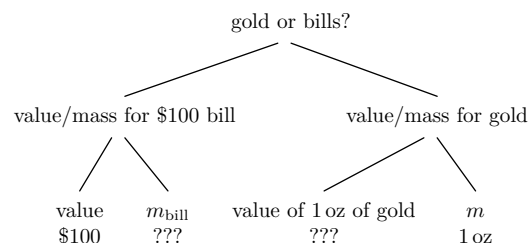
Compare this range to the range for the off-the-cuff guess $10^7 \dots 10^{12}$ barrels/yr. That range spanned a factor of 10^5 whereas the improved range spans a mere factor of 10 – thanks to divide-and-conquer reasoning.

2.6 Example 3: Gold or bills?

The chapter's final estimation example is dedicated to readers who forgo careers in the financial industry for less lucrative careers in teaching and research:

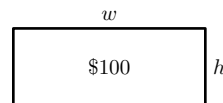
► *Having broken into a bank vault, should we take the \$100 bills or the gold?*

The answer depends partly on the ease and costs of fencing the loot – an analysis beyond the scope of this book. But within our scope is the following question: Which choice lets us carry out the most money? Our carrying capacity is limited by weight and volume. In this analysis, I assume that the lowest limit comes from weight (or mass). The mass subdivides into two subproblems – the value per mass for \$100 bills and the value per mass for gold – each of which subdivides into two subproblems:

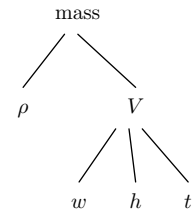


Two leaves have defined values: the value of a \$100 bill and the mass of 1 oz (1 ounce) of gold. The two other leaves need divide-and-conquer estimates. In the first round of analysis I make point estimates; in the second round, I account for uncertainty by using the plausible ranges of [Section 2.5](#).

The value of gold is now (2008), I vaguely remember, around \$800/oz. As a rough check on the value – for example, should it be \$80/oz or \$8000/oz? – here is a historical method. In 1945, at the end of World War 2, the British empire had exhausted its resources while the United States became the world's leading economic power. The gold standard was accordingly re-defined in terms of the dollar: \$35 would be the value of 1 oz of gold. Since then, inflation has probably devalued the dollar by a factor of 10 or more, so gold should be worth around \$350/oz. So my vague memory of \$800/oz seems reasonable.



For the \$100 bill, its mass breaks into density (ρ) times volume (V), and volume breaks into width (w) times height (h) times thickness (t). To estimate the height and width, I could lay down a ruler or just find a \$1 bill – all US bills are the same size – and eyeball its dimensions. A \$1 bill seems to be few inches high and 6 in wide. In metric units those dimensions are $h \sim 6$ cm and $w \sim 15$ cm. [To improve your judgment of size, first make guesses; then, if you feel unsure, check the guess using a ruler to check. With practice, your need for the ruler will decrease and your confidence and accuracy will increase.]



The thickness, alas, is not easy to estimate with eyeball or ruler. Is the thickness 1 mm or 0.1 mm or 0.01 mm? I have almost no experience with such small lengths so my eye does not help much. My ruler is calibrated in steps of 1 mm, from which I see that a piece of paper is significantly smaller than 1 mm, but I cannot easily see how much smaller.

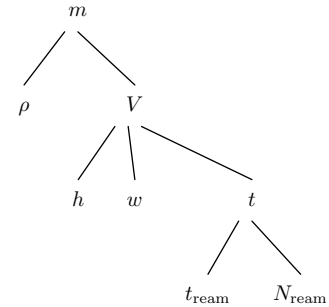
An accurate divide-and-conquer estimate, we learned in [Section 2.5](#), depends on replacing a vaguely understood quantity with accurately known quantities. Therefore to estimate the thickness accurately, I connect it to familiar quantities. Bills are made from paper, a ubiquitous substance (despite hype about the paperless office). Indeed, a ream of printer paper is just around the corner. The thickness of the ream and the number of sheets that it contains determines the thickness of one sheet:

$$t = \frac{t_{\text{ream}}}{N_{\text{ream}}}.$$

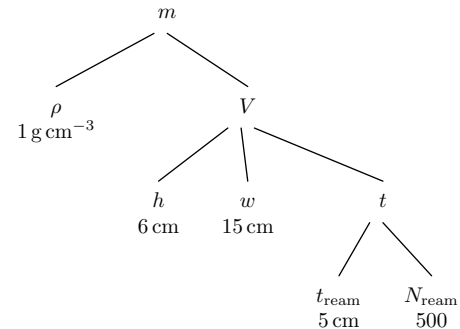
You could call this approach 'multiply and conquer'. The general lesson for accurate estimation is that values well below our experience need to be magnified, and values well above our experience need to be shrunk.

The magnification argument adds one level to the tree and replaces one leaf with two leaves on the new level. Two of the five leaf nodes are already estimated. A ream contains 500 sheets ($N_{\text{ream}} = 500$) and has a thickness of roughly 2 in or 5 cm.

► What is your estimate for ρ , the density of a \$100 bill?



The only missing leaf value is ρ , the density of a \$100 bill. Connect this value to what you already know such as the densities of familiar substances. Bills are made of paper, whose density is hard to guess directly. However, paper is made of wood, whose density is easy to guess! Wood barely floats so its density is roughly that of water: 1 g cm^{-3} . Therefore the density of a \$100 bill is roughly 1 g cm^{-3} .



Now propagate the leaf values upward. The thickness of a bill is roughly 10^{-2} cm , so the volume of a bill is roughly

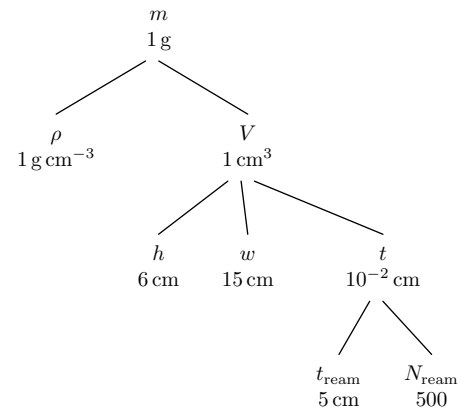
$$V \sim 6 \text{ cm} \times 15 \text{ cm} \times 10^{-2} \text{ cm} \sim 1 \text{ cm}^3.$$

The mass is therefore

$$m \sim 1 \text{ cm}^3 \times 1 \text{ g cm}^{-3} \sim 1 \text{ g}.$$

and the value per mass of a \$100 bill is therefore \$100/g. How simple!

To choose between the bills and gold, compare that value to the value per mass of gold. Unfortunately the price of gold is usually quoted in dollars per ounce rather than dollars per gram, so my vague memory of \$800/oz needs to be converted into metric units. One ounce is roughly 28 g; if the price of gold were \$840/oz, the arithmetic is simple enough to do mentally, and produces \$30/g. The exact division produces the slightly lower figure of \$28/g. Our conclusion: In the bank vault



first collect as many \$100 bills as we can carry. If we have spare capacity, collect the \$50 bills, the gold, and then the \$20 bills.

This order depends on the accuracy of the point estimates and would change if the estimates are significantly inaccurate. How accurate are they? To analyze the accuracy I will give plausible ranges for the leaf nodes and then propagate them upward to obtain plausible ranges for the value per mass of bills and gold.

Problem 2.10 Your plausible ranges

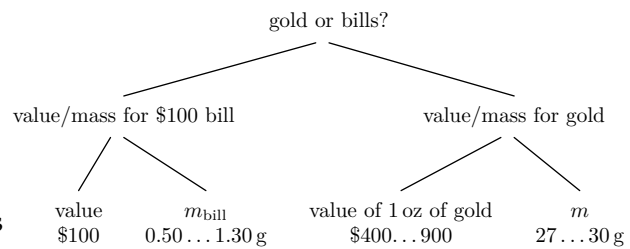
What are your plausible ranges for the five leaf quantities t_{ream} , N_{ream} , w , h , and ρ ? Propagate them upward to get plausible ranges for the interior nodes including for the root node m .

Here are my ranges along with a few notes on how I estimated a few of the ranges:

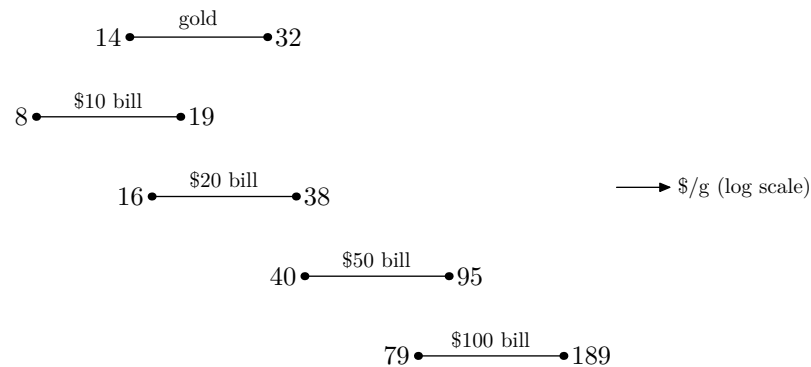
1. thickness of a ream, t_{ream} : 4 ... 6 cm.
2. number of sheets in a ream, N_{ream} : 500. I'm almost certain that I remember this value correctly, but to be certain I confirmed it by looking at a label on a fresh ream.
3. width of a bill, w : 10 ... 20 cm. A reasonable length estimate seemed to be 6 in but I could give or take a couple inches. In metric units, 4 ... 8 in becomes (roughly) 10 ... 20 cm.
4. height of a bill, h : 5 ... 7 cm.
5. density of a bill, ρ : 0.8 ... 1.2 g cm⁻³. The argument for $\rho = 1$ g cm⁻³ – that a bill is made from paper and paper is made from wood – seems reasonable. However, the processing steps may reduce or increase the density slightly.

Now propagate these ranges upward. The plausible range for t becomes 0.8 ... 1.2 · 10⁻² cm. The plausible range for the volume V becomes 0.53 ... 1.27 cm³. The plausible range for the mass m becomes 0.50 ... 1.30 g. The plausible range for the value per mass is \$79 ... 189/g (with a midpoint of \$122/g).

The next estimate is the value per mass of gold. I can be as accurate as I want in converting from ounces to grams. But I'll be lazy and try to remember the value while including uncertainty to reflect the fallibility of memory; let's say that $1\text{ oz} = 27 \dots 30\text{ g}$. This range spans only a factor of 1.1, but the value of an ounce of gold will have a wider plausible range (except for those who often deal with financial markets). My range is $\$400 \dots \900 . The mass and value ranges combine to give $\$14 \dots \$32/\text{g}$ as the range for gold.



Here is a picture comparing the range for gold with the ranges for US currency denominations:



Looking at the locations of these ranges and overlaps among them, I am confident that the \$100 bills are worth more (per mass) than gold. I am reasonably confident that \$50 bills are worth more than gold, undecided about \$20 bills, and reasonably confident that \$10 bills are worth less than gold.

2.7 Example 4: The UNIX philosophy

The preceding examples illustrate how divide and conquer enables accurate estimates. An example remote from estimation – the design principles of the UNIX operating system – illustrates the generality of this tool.

UNIX and its close cousins such as GNU/Linux operate devices as small as cellular telephones and as large as supercomputers cooled by liquid nitrogen. They constitute the world's most portable operating system. Its success derives not from marketing – the most successful variant, GNU/Linux,

is free software and owned by no corporation – but rather from outstanding design principles.

These principles are the subject of *The UNIX Philosophy* [14], a valuable book for anyone interested in how to design large systems. The author isolates nine tenets of the UNIX philosophy, of which four – those with comments in the following list – incorporate or enable divide-and-conquer reasoning:

1. Small is beautiful. In estimation problems, divide and conquer works by replacing quantities about which one knows little with quantities about which one knows more (Section 2.5). Similarly, hard computational problems – for example, building a searchable database of all emails or web pages – can often be solved by breaking them into small, well-understood tasks. Small programs, being easy to understand and use, therefore make good leaf nodes in a divide-and-conquer tree (Section 2.3).
2. Make each program do one thing well. A program doing one task – only spell-checking rather than all of word processing – is easier to understand, to debug, and to use. One-task programs therefore make good leaf nodes in a divide-and-conquer trees.
3. Build a prototype as soon as possible.
4. Choose portability over efficiency.
5. Store data in flat text files.
6. Use software leverage to your advantage.
7. Use shell scripts to increase leverage and portability.
8. Avoid captive user interfaces. Such interfaces are typical in programs for solving complex tasks, for example managing email or writing documents. These monolithic solutions, besides being large and hard to debug, hold the user captive in their pre-designed set of operations.

In contrast, UNIX programmers typically solve complex tasks by dividing them into smaller tasks and conquering those tasks with simple programs. The user can adapt and remix these simple programs to solve problems unanticipated by the programmer.

9. Make every program a filter. A filter, in programming parlance, takes input data, processes it, and produces new data. A filter combines easily with another filter, with the output from one filter becoming the input for the next filter. Filters therefore make good leaves in a divide-and-conquer tree.

As examples of these principles, here are two UNIX programs, each a small filter doing one task well:

- **head**: prints the first lines of the input. For example, **head** invoked as **head -15** prints the first 15 lines.
- **tail**: prints the last lines of the input. For example, **tail** invoked as **tail -15** prints the last 15 lines.

► *How can you use these building blocks to print the 23rd line of a file?*

This problem subdivides into two parts: (1) print the first 23 lines, then (2) print the last line of those first 23 lines. The first subproblem is solved with the filter **head -23**. The second subproblem is solved with the filter **tail -1**.

The remaining problem is how to hand the second filter the output of the first filter – in other words how to combine the leaves of the tree. In estimation problems, we usually multiply the leaf values, so the combinator is usually the multiplication operator. In UNIX, the combinator is the **pipe**. Just as a plumber's pipe connects the output of one object, such as a sink, to the input of another object (often a larger pipe system), a UNIX pipe connects the output of one program to the input of another program.

The pipe syntax is the vertical bar. Therefore, the following **pipeline** prints the 23rd line from its input:

```
head -23 | tail -1
```

But where does the system get the input? There are several ways to tell it where to look:

1. Use the pipeline unchanged. Then **head** reads its input from the keyboard. A UNIX convention – not a requirement, but a habit followed by most programs – is that, unless an input file is specified, programs read from the so-called standard input stream, usually the keyboard. The pipeline

```
head -23 | tail -1
```

therefore reads lines typed at the keyboard, prints the 23rd line, and exits (even if the user is still typing).

2. Tell **head** to read its input from a file – for example from an English dictionary. On my GNU/Linux computer, the English dictionary is the file `/usr/share/dict/words`. It contains one word per line, so the following pipeline prints the 23rd word from the dictionary:

```
head -23 /usr/share/dict/words | tail -1
```

3. Let `head` read from its standard input, but connect the standard input to a file:

```
head -23 < /usr/share/dict/words | tail -1
```

The `<` operator tells the UNIX command interpreter to connect the file `/usr/share/dict/words` to the input of `head`. The system tricks `head` into thinking its reading from the keyboard, but the input comes from the file – without requiring any change in the program!

4. Use the `cat` program to achieve the same effect as the preceding method. The `cat` program copies its input file(s) to the output. This extended pipeline therefore has the same effect as the preceding method:

```
cat /usr/share/dict/words | head -23 | tail -1
```

This longer pipeline is slightly less efficient than using the redirection operator. The pipeline requires an extra program (`cat`) copying its input to its output, whereas the redirection operator lets the lower level of the UNIX system achieve the same effect (replumbing the input) without the gratuitous copy.

As practice, let's use the UNIX approach to divide and conquer a search problem:

- *Imagine a dictionary of English alphabetized from right to left instead of the usual left to right. In other words, the dictionary begins with words that end in 'a'. In that dictionary, what word immediately follows *trivia*?*

This whimsical problem is drawn from a scavenger hunt [29] created by the computer scientist Donald Knuth, whose many accomplishments include the $\text{\TeX}$ typesetting system used to produce this book.

The UNIX approach divides the problem into two parts:

1. Make a dictionary alphabetized from right to left.
2. Print the line following 'trivia'.

The first problem subdivides into three parts:

1. Reverse each line of a regular dictionary.
2. Alphabetize (sort) the reversed dictionary.
3. Reverse each line to undo the effect of **step 1**.

The second part is solved by the UNIX utility `sort`. For the first and third parts, perhaps a solution is provided by an item in UNIX toolbox. However, it would take a long time to thumb through the toolbox hoping to get lucky: My computer tells me that it has over 8000 system programs.

Fortunately, the UNIX utility `man` does the work for us. `man` with the `-k` option, with the 'k' standing for keyword, lists programs with a specified keyword in their name or one-line description. On my laptop, `man -k reverse` says:

```
$ man -k reverse
col (1)          - filter reverse line feeds from input
git-rev-list (1) - Lists commit objects in reverse chronological order
rev (1)          - reverse lines of a file or files
tac (1)          - concatenate and print files in reverse
xxd (1)          - make a hexdump or do the reverse.
```

Understanding the free-form English text in the one-line descriptions is not a strength of current computers, so I leaf through this list by hand – but it contains only five items rather than 8000. Looking at the list, I spot `rev` as a filter that reverses each line of its input.

► *How do you use `rev` and `sort` to alphabetize the dictionary from right to left?*

Therefore the following pipeline alphabetizes the dictionary from right to left:

```
rev < /usr/share/dict/words | sort | rev
```

The second problem – finding the line after 'trivia' – is a task for the pattern-searching utility `grep`. If you had not known about `grep`, you might find it by asking the system for help with `man -k pattern`. Among the short list is

```
grep (1)          - print lines matching a pattern
```

In its simplest usage, `grep` prints every input line that matches a specified pattern. For example,

```
grep 'trivia' < /usr/share/dict/words
```

prints all lines that contain `trivia`. Besides `trivia` itself, the output includes `trivial`, `nontrivial`, `trivializes`, and similar words. To require that the word match `trivia` with no characters before or after it, give `grep` this pattern:

```
grep '^trivia$' < /usr/share/dict/words
```

The patterns are **regular expressions**. Their syntax can become arcane but their important features are simple. The `^` character matches the beginning of the line, and the `$` character matches the end of the line. So the pattern `trivia$` selects only lines that contain exactly the text `trivia`.

- *This invocation of `grep`, with the special characters anchoring the beginning and ending of the lines, simply prints the word that I specified. How could such an invocation be useful?*

That invocation of `grep` tells us only that `trivia` is in the dictionary. So it is useful for checking spelling – the solution to a problem, but not to our problem of finding the word that follows `trivia`. However, Invoked with the `-A` option, `grep` prints lines following each matching line. For example,

```
grep -A 3 '^trivia$' < /usr/share/dict/words
```

will print `'trivia'` and the three lines (words) that follow it.

```
trivia
trivial
trivialities
triviality
```

To print only the word after `'trivia'` but not `'trivia'` itself, use `tail`:

```
grep -A 1 '^trivia$' < /usr/share/dict/words | tail -1
```

These small solutions combine to solve the scavenger-hunt problem:

```
rev </usr/share/dict/words | sort | rev | grep -A 1 '^trivia$' | tail -1
```

- *Try it on a local UNIX or GNU/Linux system. How well does it work?*

Alas, on my system, the pipeline fails with the error

```
rev: stdin: Invalid or incomplete multibyte or wide character
```

The `rev` program is complaining that it does not understand a character in the dictionary. `rev` is from the old, ASCII-only days of UNIX, when each character was limited to one byte; the dictionary, however, is a modern one and includes Unicode characters to represent the accented letters prevalent in European languages.

To solve this unexpected problem, I clean the dictionary before passing it to `rev`. The cleaning program is again the filter `grep` told to allow through only pure ASCII lines. The following command filters the dictionary to contain words made only of unaccented, lowercase letters.

```
grep '^[a-z]*$' < /usr/share/dict/words
```

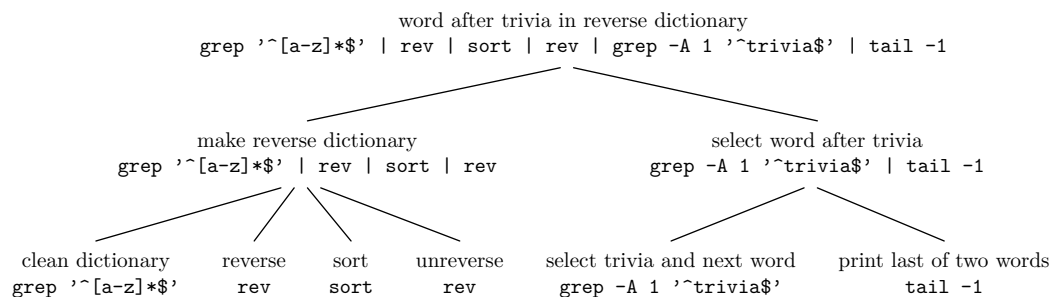
This pattern uses the most important features of the regular-expression language. The `^` and `$` characters have been explained in the preceding examples. The `[a-z]` notation means ‘match any character in the range a to z – i.e. match any lowercase letter.’ The `*` character means ‘match zero or more occurrences of the preceding regular expression’. So `^[a-z]*$` matches any line that contains only lowercase letters – no Unicode characters allowed.

The full pipeline is

```
grep '^[a-z]*$' < /usr/share/dict/words \
| rev | sort | rev \
| grep -A 1 '^trivia$' | tail -1
```

where the backslashes at the end of the lines tell the shell to continue reading the command beyond the end of that line.

The tree representing this solution is



Running the pipeline produces produces ‘alluvia’.

Problem 2.11 Angry

In the reverse-alphabetized dictionary, what word follows angry?

Although solving this problem won't save the world, it illustrates how divide-and-conquer reasoning is built into the design of UNIX. In short, divide and conquer is a ubiquitous tool useful for estimating difficult quantities or for designing large, successful systems.

Main messages

This chapter has tried to illustrate these messages:

1. Divide large, difficult problems into smaller, easier ones.
2. Accuracy comes from subdividing until you reach problems about which you know more or can easily solve.
3. Trees compactly represent divide-and-conquer reasoning.
4. Divide-and-conquer reasoning is a cross-domain tool, useful in text processing, engineering estimates, and even economics.

By breaking hard problems into comprehensible units, the divide-and-conquer tool helps us organize complexity. The next chapter examines its cousin abstraction, another way to organize complexity.

Problem 2.12 Air mass

Estimate the mass of air in the 6.055J/2.038J classroom and explain your estimate with a tree. If you have not seen the classroom yet, then make more effort to come to lecture (!); meanwhile pictures of the classroom are linked from the course website.

Problem 2.13 747

Estimate the mass of a full 747 jumbo jet, explaining your estimate using a tree. Then compare with data online. We'll use this value later this semester for estimating the energy costs of flying.

Problem 2.14 Random walks and accuracy of divide and conquer

Use a coin, a random-number function (in whatever programming language you like), or a table of reasonably random numbers to do the following experiments or their equivalent.

The first experiment:

1. Flip a coin 25 times. For each heads move right one step; for each tails, move left one step. At the end of the 25 steps, record your position as a number between -25 and 25 .
2. Repeat the above procedure four times (i.e. three more times), and mark your four ending positions on a number line.

The second experiment:

1. Flip a coin once. For heads, move right 25 steps; for tails, move left 25 steps.
2. Repeat the above procedure four times (i.e. three more times), and mark your four ending positions on a second number line.

Compare the marks on the two number lines, and explain the relation between this data and the model from lecture for why divide and conquer often reduces errors.

Problem 2.15 Fish tank

Estimate the mass of a typical home fish tank (filled with water and fish): a useful exercise before you help a friend move who has a fish tank.

Problem 2.16 Bandwidth

Estimate the bandwidth (bits/s) of a 747 crossing the Atlantic filled with CDROM's.

Problem 2.17 Repainting MIT

Estimate the cost to repaint all indoor walls in the main MIT classroom buildings. [with thanks to D. Zurovcik]

Problem 2.18 Explain a UNIX pipeline

What does this pipeline do?

```
ls -t | head | tac
```

[Hint: If you are not familiar with UNIX commands, use the `man` command on Athena or on any nearby UNIX or GNU/Linux system.]

Problem 2.19 Design a UNIX pipeline

Make a pipeline that prints the ten most common words in the input stream, along with how many times each word occurs. They should be printed in order from the most frequent to the less frequent words. [Hint: First translate any non-alphabetic character into a newline. Useful utilities include `tr` and `uniq`.]

Chapter 3

Abstraction

Our first tool, divide and conquer, breaks enigmas into manageable problems. These leaf nodes are manageable partly because they are conceptually simple; the length of a classical symphony, roughly one hour, is a simple concept compared to the data capacity of a CDROM. Successful leaf nodes are manageable also because they are familiar. The length of a symphony, for example, might be familiar from attending a classical concert. A concert is typically 2.5 hours with a half-hour intermission (interval) in the middle, leaving one-hour blocks at the start or end for a full symphony.

Familiarity is the sibling of reuse. Successful divide-and-conquer reasoning breaks a problem not just into parts but into reusable parts. Discovering and constructing such parts is the purpose of **abstraction** – the second tool for organizing complexity. Abstraction is, according to the *Oxford English Dictionary* [24]:

The act or process of separating in thought, *of considering a thing independently of its associations*; or a substance independently of its attributes; or an attribute or quality independently of the substance to which it belongs.
[my italics]

Abstraction thereby generates new ideas and new units of thought.

The tools taught in this book are themselves fruits of abstraction. For example, many estimation problems were solved by dividing the problem into small, manageable parts. This pattern needed a name – divide and conquer. The other tools, even abstraction, are similarly the fruits of abstraction.

3.1 Reusability

The most important characteristic of abstraction is reusability. As Abelson and Sussman [1, s. 1.1.8] eloquently describe:

The importance of this decomposition strategy is not simply that one is dividing the program into parts. After all, we could take any large program and divide it into parts – the first ten lines, the next ten lines, the next ten lines, and so on. Rather, it is crucial that each procedure accomplishes an identifiable task that can be used as a module in defining other procedures.

To understand what makes a useful, reusable abstraction, let's examine a weak, barely reusable abstraction and compare its features with the features of a useful abstraction. So, imagine that the designers of UNIX had noticed that users often needed to count how often each word appears in a document, listing the most frequent words first (along with their frequencies). One solution is to provide a special utility called `sortedwordfreq`. This utility, however, cannot solve any other problem.

As an improvement, the problem could be broken into three steps:

1. break the document into words, one per line.
2. count how often each word appears
3. sort the frequency list by frequency

UNIX could provide three command-line programs, one for each step, and the user would connect the programs with pipes:

```
break_into_words < file.txt | wordfreq | sort_frequency_list
```

where `sort_frequency_list` sorts a list such as

```
34 an
273 the
12 where
23 none
```

to produce

```
273 the
34 an
23 none
12 where
```

This approach is an improvement on the monolithic solution because one of the three pieces might be used in solving a different problem.

The actual solution using the UNIX tools is even more reusable. Rather than provide a special program to break a document into words, one per line,

UNIX provides a utility called `tr`. It translates characters into other characters. So ask it to translate all non-alphabetic characters into newline characters, and then to squeeze repeated newline characters into one newline character. That command is

```
tr -s -c 'a-zA-Z' '\012'
```

The `-s` option tells `tr` to squeeze repeated newlines into one newline. The `-c` option tells `tr` to invert (complement) the following character set (the upper- and lowercase alphabet). It is simpler to specify the non-alphabetic characters by what they are not than by what they are. So this invocation of `tr` turns any non-alphabetic character into a newline, then squeezes repeated newlines into one newline.

It turns the first sentence of this paragraph into the following list of words, one per line:

```
The
actual
solution
using
the
Unix
tools
is
even
more
reusable
Rather
than
provide
a
special
program
to
break
a
document
into
words
```

The next step is to count how often each word appears. Perhaps UNIX provides a program called `count` that performs this task? Such a program

would have to look through the entire list and accumulating counts. It is simpler first to sort the list. Then identical words appear in clumps, which means that the counting program need not scan the entire list. Instead it can consider one clump at a time. The sorting step is accomplished by the familiar program `sort`. The clump counting is accomplished by a new program, `uniq` with the `-c` option.

Here is the result of taking the text of Gibbon's *Decline and Fall of the Roman Empire* (volume 1) and feeding it to the pipeline: first taking out all punctuation and turning it into a list of words, one per line; then sorting the list; then counting the clumps. The result is:

```
$ tr -cs 'a-zA-Z' '\012' < decline.txt | sort | uniq -c
  4452 a
   233 A
     2 Aaron
     9 ab
     1 Ab
     8 abandon
    30 abandoned
     1 Abandoning
     1 abandonment
     3 Abate
    ...
```

These are not the 10 most common words! Rather, they are the 10 alphabetically earliest words (along with their counts). To find the most common words, sort this output numerically by adding `sort -nr` to the end of the pipeline:

```
$ tr -cs 'a-zA-Z' '\012' < decline.txt | sort | uniq -c |
sort -nr
24241 the
17920 of
 9097 and
 5951 to
 4452 a
 3869 in
 3171 was
 2904 his
 2737 by
 2711 The
    ...
```

We're almost there! But a problem is the appearance of 'The' on a separate line. We forgot about uppercase versus lowercase. So let's use `tr` one more time (what a useful abstraction), to turn uppercase into lowercase:

```
$ tr -cs 'a-zA-Z' '\012' < decline.txt | tr 'A-Z' 'a-z' |
sort | uniq -c | sort -nr
  26960 the
  18099 of
   9168 and
   6050 to
   4685 a
   4217 in
   3171 was
   3081 his
   2815 by
   2396 that
    ...
```

The new count for 'the' is 29690. But the count for 'the' together the count for 'The' give a count of $24241 + 2711 = 26952$. What accounts for the discrepancy between 26952 and 26960? Let's ask UNIX to tell us about all forms of 'the' that showed up. To do so, use `grep` to match only lines reporting counts for 'the' or one its mixed-case variants. The `-i` option to `grep` tells `grep` not to care about upper versus lowercase. The pattern for `grep` to look for is then a space followed by 'the' followed by the end of line (`$` in `grep` notation). So the pipeline with its output is:

```
$ tr -cs 'a-zA-Z' '\012' < decline.txt | sort | uniq -c |
grep -i ' the$'
  24241 the
   2711 The
     8 THE
```

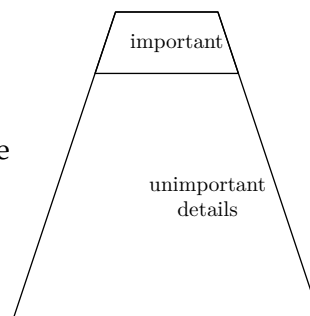
Ah, so there were eight appearances of 'THE' – which accounts for the discrepancy between 26952 and 26960.

3.2 Notation and hierarchy

Good notation promotes good thinking.

An abstraction divides a phenomenon into two pieces: the small piece above the water about which I want to think, and the large piece below the water about which I want to avoid thinking. By pushing irrelevant details below the water while keeping relevant information above the water, you keep the relevant details prominent in your mind. Good abstractions, by making useful units, make us smart.

Here are two examples: computers and fluid mechanics.



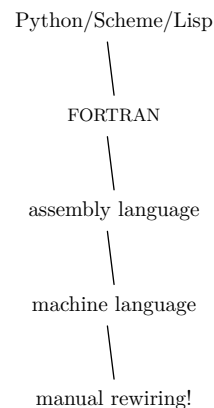
Early computers were programmed in machine language, a sequence of numbers understood directly by the hardware. It improved on the previous method of programming, plugging and unplugging wires, but it was a cumbersome abstraction. Building upon it, the next advance was assembly language; instead of the magic numbers of machine language, assembly language allowed the programmer to use symbolic names such as LOAD and STORE. The next abstraction in programming languages was FORTRAN [4], short for **formula translation**. Instead of a complicated sequence of assembly-language instructions, the programmer could write

```
a = b * c
```

letting the computer figure out how to load, multiply, and store the values. The power of this advance is illustrated by the longevity of FORTRAN: Proposed in 1953 and implemented in 1957, it has been in wide use for over 50 years. FORTRAN did not simplify all common tasks, however. String processing, at least in early versions, had to be programmed explicitly. Whereas languages such as LISP and Python make it easy by abstracting away the programming details. The following Python statement would require many lines of FORTRAN:

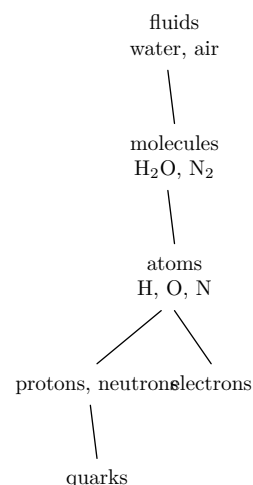
```
filename = directory + '/' + basename + '.txt'
```

The abstraction tower does not stop with Python. Prolog, a high-level language, incorporates search algorithms as basic elements of the language. And people dream of a do-what-I-mean language where the programming just describes what he or she wants done, and the computer figures out how to do it.

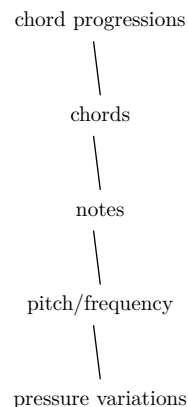


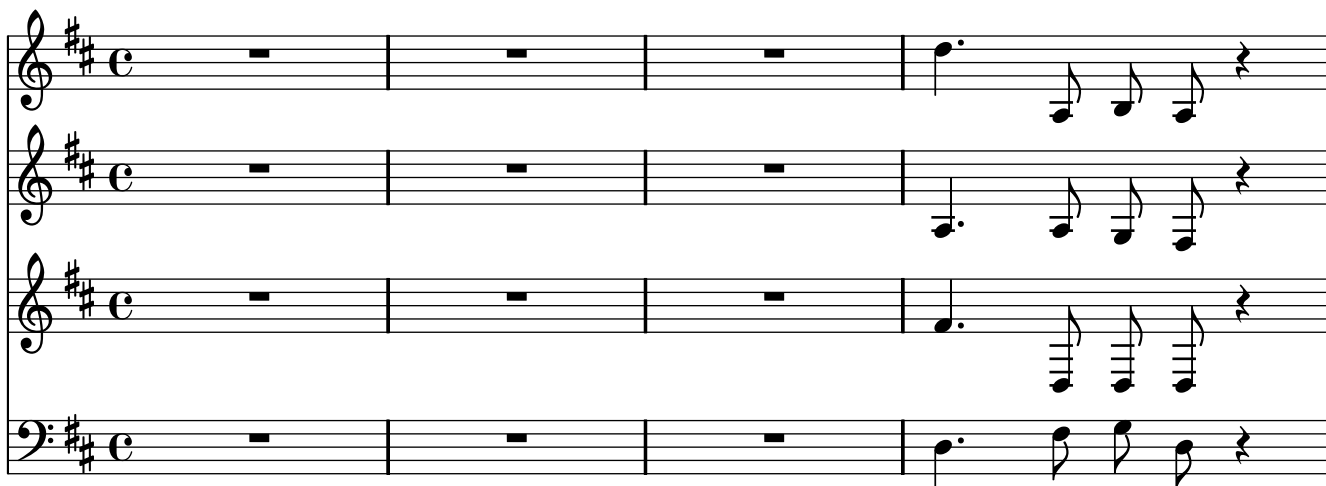
Another high-level abstraction is fluids. At the bottom of that tower are the actors of fundamental physics: quarks and electrons. Quarks combine to build protons and neutrons. Protons, neutrons, and electrons combine to build atoms. Atoms combine to build molecules. And large collections of molecules act – under some conditions – like a fluid.

No abstraction tower stands alone. Layers usually exist below a tower. Beneath machine language, for example, into which high-level languages like Python are translated, lies a tower centered on circuits. Machine language is interpreted by the central-processing unit (CPU). The CPU is designed using logic gates. Logic gates are designed using circuit elements such as transistors, resistors, and capacitors. These circuit elements are themselves abstractions based on the flow of electrons in semiconductors and metals. Semiconductors and metals are themselves abstractions for how large collections of atoms and molecules behave. Underneath that layer lies a tower running from quarks and electrons at the bottom to atoms and molecules at the top.



Layers usually exist above a tower. Music, for example, sits on the abstraction tower for fluids. Music itself is a sound wave, described completely by the pressure as a function of space and time. Pressure and pressure variations are abstractions building on fluids. When the pressure variations fluctuate in a regular pattern, we assign them a frequency or, what is almost the same idea, a pitch – such as 550 Hz. Certain pitches we assign to musical notes such as middle C.





They are the opening measures of the ‘Hallelujah chorus’ from Handel’s *Messiah*. As you look at it, remember – for a moment – the many layers of acoustics that it hides. Now forget those layers, and instead climb the abstraction ladder. As a first step, let’s look at the first notes sung by the choir. They are D, F, and A, with the D repeated in the highest (soprano) line. The D, F, and A notes form a new abstraction: a D major triad. That abstraction ignores the difference between a low D and a high D. The piece is written in the key of D major, so the D major triad is the fundamental triad, also called the I triad. Subsequent stacks of notes produce a sequence of chords with higher-level, longer-term patterns. A formal method of analysis – a recipe for abstraction – is Heinrich Schenker’s hierarchical approach to analysis [5].

Note the importance, again, of reusability. If the notation or, more generally, the representation is not reusable, it is worth little. The musical notation is highly reusable. Millions of works are written in it, using and reusing the same symbols.

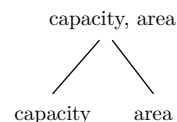
3.2.1 Leakiness

Good abstractions tend to break free from their moorings and develop a life of their own. We then forget that almost every abstraction is leaky, that abstraction barriers are rarely impermeable. For example, programming languages such as Python allow, even encourage the programmer to treat strings as fundamental objects that can be combined as simply as numbers

can be added. However, if the strings are long, the computer may run out of memory. (Even that statement hides a leaky abstraction tower, because most computers implement the abstraction of virtual memory so that many programs, from many users, can share the same physical memory without contention.) A lower-level language such as assembly language would have forced the programmer to think about memory allocation. Similarly, as a collection of molecules becomes sparse, the fluid abstraction starts to fail. Piano sheet music uses such a good notation that we hardly remember that it is yet another abstraction, let alone that it is leaky. And it has to be leaky, otherwise a performance in the King's College Chapel in Cambridge, with its beautiful acoustics, would be identical to a performance in a thunderstorm in the wind; or a performance by Glenn Gould and Vladimir Horowitz, two great pianists of the 20th century, would be no different than a performance by Sanjoy Mahajan (an average piano player who someday would like to be a pianist). Music indeed sounds differently depending on where you sit in the room, on who sings or plays the notes, and even on what the audience is wearing. All that variation – all that complexity – is ignored by the usual notation. In other words, the usual notation is leaky. So, abstractions are essential to living and thinking but they have limits beyond which they are not applicable.

3.3 Example: Minilanguages

I made the tree diagram in the margin, copied the beginning of [Section 2.3](#), using the MetaPost graphics language. This language has a package – a set of useful functions – for making boxes. Here is the program:



```

% specify the texts
boxit.root(btex capacity, area etex);
boxit.capacity(btex capacity etex);
boxit.area(btex area etex);
% specify their relative positions
ypart(capacity.n-area.n) = 0;
xpart(area.w-capacity.e) = 10pt;
root.s - 0.5[capacity.ne,area.nw] = (0,20pt);
% place (draw) the texts without borders
drawunboxed(root, capacity, area);
% connect root with its two children
draw root.s shifted (-5pt,0) -- capacity.n;
draw root.s -- area.n;

```

Never mind the full syntax of the language. Notice that it takes such a long program to make such a simple diagram! Notice also that the operations are repetitive. For example, the direct children of the root have the same horizontal position; if there were grandchildren, they would have the same horizontal position, different from the position of the children. And so on. These repeating motifs suggest that this program is written at the wrong level of abstraction.

After using the boxes package to program a few tree diagrams with more elements, I finally took my own medicine, and wrote a **minilanguage** specialized for drawing tree diagrams. The preceding diagram is specified by these three lines:

```
capacity, area
  capacity
    area
```

A script turns those three lines into code using the boxes package.

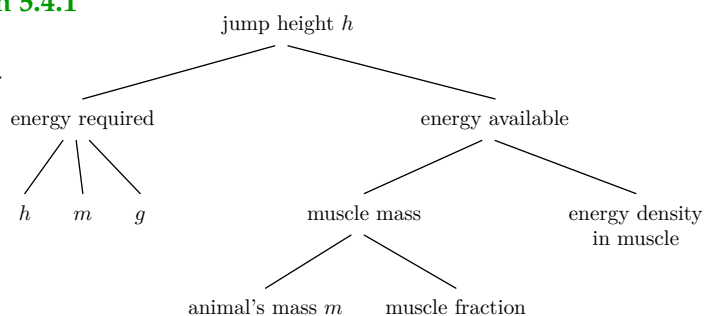
The tree minilanguage makes constructing tree diagrams so easy that I happily created many of them to explain divide-and-conquer reasoning in [Chapter 2](#). The tree minilanguage does not solve a great problem of the world, but the idea of minilanguages is a great idea. In computation, an excellent introduction is Eric Raymond's *The Art of UNIX Programming* [28, Chapter 8], which begins with a quote from Bertrand Russell:

A good notation has a subtlety and suggestiveness which makes it almost seem like a live teacher.

Here is a figure from [Section 5.4.1](#)

later in the book. Here is its program in the tree minilanguage:

```
jump height $h$
  energy required
    $h$
    $m$
    $g$
  energy available
    muscle mass
      animal's mass $m$
      muscle fraction
    energy density|in muscle
```



This simple code – simple to understand and simple to write – expands into 34 lines of tedious, error-prone MetaPost boxes code (not shown here to avoid boring you). The moral is: Let a computer, which rarely makes errors, do the translation, and do your thinking using the higher-level abstractions.

3.4 Example: Operators

The next abstraction is two levels more abstract than ordinary numbers. Ordinary numbers are

Operators turn functions into functions. The space of functions is itself vast and complex, so operators are complex beasts. Ignoring most of that complexity makes operators act like ordinary numbers. Although this abstraction is leaky, it leaks so rarely that we can figure out a lot by adopting it and charging ahead fearlessly. ‘Be approximately right rather than exactly wrong’ (attributed to John Tukey and John Maynard Keynes).

3.4.1 Derivative operator

A familiar operator is the derivative. Here is a differential equation for the motion of a damped spring, in a suitable system of units:

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + x = 0,$$

where x is dimensionless position, and t is dimensionless time. Imagine x as the amplitude divided by the initial amplitude; and t as the time multiplied by the frequency (so it is radians of oscillation). The dx/dt term represents the friction, and its plus sign indicates that friction dissipates the system’s energy. A useful shorthand for the d/dt is the operator D . It is an operator because it operates on an object – here a function – and returns another object. Using D , the spring’s equation becomes

$$D^2x(t) + 3Dx(t) + x(t) = 0.$$

The tricky step is replacing d^2x/dt^2 by D^2x , as follows:

$$D^2x = D(Dx) = D\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}.$$

The analogy comes in solving the equation. Pretend that D is a number, and do to it what you would do with numbers. For example, factor the equation. First, factor out the $x(t)$ or x , then factor the polynomial in D :

$$(D^2 + 3D + 1)x = (D + 2)(D + 1)x = 0.$$

This equation is satisfied if either $(D + 1)x = 0$ or $(D + 2)x = 0$. The first equation written in normal form, becomes

$$(D + 1)x = \frac{dx}{dt} + x = 0,$$

or $x = e^{-t}$ (give or take a constant). The second equation becomes

$$(D + 2)x = \frac{dx}{dt} + 2x = 0,$$

or $x = e^{-2t}$. So the equation has two solutions: $x = e^{-t}$ or e^{-2t} .

The example above introduced D and its square, D^2 , the second derivative. You can do more with the operator D . You can cube it, take its logarithm, its reciprocal, and even its exponential. Let's look at the exponential e^D . It has a power series:

$$e^D = 1 + D + \frac{1}{2}D^2 + \frac{1}{6}D^3 + \dots$$

That's a new operator. Let's see what it does by letting it operating on a few functions. For example, apply it to $x = t$:

$$(1 + D + D^2/2 + \dots)t = t + 1 + 0 = t + 1.$$

And to $x = t^2$:

$$(1 + D + D^2/2 + D^3/6 + \dots)t^2 = t^2 + 2t + 1 + 0 = (t + 1)^2.$$

And to $x = t^3$:

$$(1 + D + D^2/2 + D^3/6 + D^4/24 + \dots)t^3 = t^3 + 3t^2 + 3t + 1 + 0 = (t + 1)^3.$$

It seems like, from these simple functions (extreme cases again), that $e^D x(t) = x(t + 1)$. You can show that for any power $x = t^n$, that

$$e^D t^n = (t + 1)^n.$$

Since any function can, pretty much, be written as a power series, and e^D is a linear operator, it acts the same on any function, not just on the powers. So e^D is the successor function: It replaces $x(t)$ by $x(t + 1)$.

3.4.2 Successor operator

Now that we know how to represent the successor operator in terms of derivatives, let's give it a name: S , and use it. It is useful in finding sums and evaluating derivatives. Let's first use it for evaluating derivatives. Suppose you sample a function and want to compute its derivative at one of the points.

$$D = \ln(S) = \ln(1 + (S - 1)) = (S - 1) + (S - 1)^2/2 + \dots$$

3.4.3 Euler-MacLaurin Summation

Suppose you have a function $f(n)$ and you want to find the sum $\sum f(k)$. Never mind the limits for now. It's a new function of n , so summation, like integration, takes a function and produces another function. It is an operator, $\sum$. Let's figure out how to represent it in terms of familiar operators. To keep it all straight, let's get the limits right. Let's define it this way:

$$F(n) = (\sum f)(n) = \sum_{-\infty}^n f(k).$$

So $f(n)$ goes into the maw of the summation operator and comes out as $F(n)$. Look at $SF(n)$. On the one hand, it is $F(n + 1)$, since that's what S does. On the other hand, S is, by analogy, just a number, so let's swap it inside the definition of $F(n)$:

$$SF(n) = (\sum Sf)(n) = \sum_{-\infty}^n f(k + 1).$$

The sum on the right is $F(n) + f(n + 1)$, so

$$SF(n) - F(n) = f(n + 1).$$

Now factor the $F(n)$ out, and replace it by $\sum f$:

$$((S - 1) \sum f)(n) = f(n + 1).$$

So $(S - 1) \sum = S$, which is an implicit equation for the operator $\sum$ in terms of S . Now let's solve it:

$$\sum = \frac{S}{S - 1} = \frac{1}{1 - S^{-1}}.$$

Since $S = e^D$, this becomes

$$\sum = \frac{1}{1 - e^{-D}}.$$

Again, remember that for our purposes D is just a number, so find the power series of the function on the right:

$$\sum = D^{-1} + \frac{1}{2} + \frac{1}{12}D - \frac{1}{720}D^3 + \dots.$$

The coefficients do not have an obvious pattern. But they are the Bernoulli numbers. Anyway, let's look at the terms one by one to see what the mean. First is D^{-1} , which is the inverse of D . Since D is the derivative operator, its inverse is the integral operator. So the first approximation to the sum is the integral – what we know from first-year calculus.

The first correction is $1/2$. Huh? Are we supposed to add $1/2$ to the integral, no matter what function we are summing? That cannot be right. And it isn't. The $1/2$ is one piece of an operator, and the whole sum is applied to a function. Let's take it in slow motion:

$$\sum f(n) = \int^n f(k) dk + \frac{1}{2}f(n) + \dots.$$

So the first correction is one-half of the final term $f(n)$.

Problem 3.1 Pictorial explanation

Find a pictorial explanation for the $f(n)/2$ term in $\sum f(n)$.

3.4.4 Euler sum

Let's improve the estimate for the Euler sum $\sum_1^\infty n^{-2}$. The first term is 1, the result of integrating. The second term is $1/2$, the result of $f(1)/2$. The third term is $1/6$, the result of $D/12$ applied to n^{-2} . So:

$$\sum_1^\infty n^{-2} \approx 1 + \frac{1}{2} + \frac{1}{6} = 1.666\dots$$

The true value is $1.644\dots$, so we're close. The fourth term gives a correction of $-1/30$. So the new value is $1.633\dots$. The approximation gets better and better!

Let's see where the $\pi^2/6$ comes from, by using analogy at a key step. Look at the function $\sin x$. That intersects the x -axis at $\pm n\pi$, where $n = 0, 1, 2, \dots$

Let's get rid of the π by looking at the function $\sin \pi x$, which has roots at $\pm nx$. Now, $\sin z$ is an **entire function**: It has no infinities – no **poles** – for any z , even for complex z . Polynomials also have no poles. An entire function is analogous to a polynomial: It is an infinite-degree polynomial. Others are e^z and $\sinh z$. Why is the analogy useful? Because your knowledge from the source system helps you generate ideas to use in the destination system. Polynomials are characterized by their zeros, so maybe entire functions are as well. For polynomials, that characterization is done by factoring them. So let's factor entire functions too.

How does $\sin \pi z$ factor? We already have a good idea.

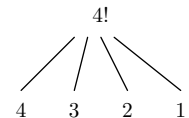
As we'll see in a later chapter, rational functions generalize to what are called **meromorphic functions** in complex analysis: functions with zeros and poles.

3.5 Example: Recursion

Sometimes you make a minilanguage to solve just one problem. The minilanguage or abstraction is reusable, and is reused multiple times in solving that problem. Recursion is an example of this use of abstraction.

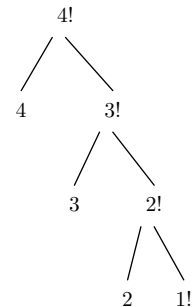
A classic example of recursion is computing $n!$. Here is a non-recursive definition of factorial:

$$n! \equiv n \times (n - 1) \times (n - 2) \times \cdots \times 1.$$



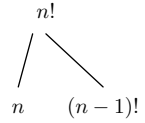
The tree illustrates how to compute $4!$ using this definition: You multiply 4, 3, 2, and 1.

Then I have a great insight: You notice that $3 \times 2 \times 1$ is also $3!$, which is $3 \times 2!$, and so on. This realization turns the flat, seemingly unstructured tree into a tree with a pattern.



Here is the pattern (when $n > 1$). The Python code that implements this idea is

```
def fact(n):
    if n == 1:
        return 1
    else:
        return n * fact(n-1)
```



The tree approach, and the corresponding code, divides the computation of $n!$ into three parts:

1. digging up n , which is easy;
2. computing $(n - 1)!$, whose details I don't care about because I know how to compute factorial; and
3. multiplying n and $(n - 1)!$, which is easy.

The abstraction is reusable: It works not just for $4!$ but for $n!$ where n is any positive integer.

In keeping with the principle of telling lies first, and removing them later, I confess that multiplying n and $(n - 1)!$ is not easy when n is large because then $(n - 1)!$ is gigantic, larger than what the central processing unit of my computer can handle in its hardware. As a second example of recursion, I describe a quick way to multiply very large integers. For simplicity, I instead describe a way to square very large integers. In the problems, you get to generalize the method to multiplication of two different integers.

First, I square 35 using the common method, then using a fast method. I use base 10 and small examples to illustrate the methods.

Okay, the common method:

$$35^2 = (3 \times 10 + 5)^2 = (3 \times 10)^2 + 2 \times 3 \times 10 \times 5 + 5^2.$$

In a pictorial abstraction, where $3|5$ represents 35 and in general $x|y$ represents $10x + y$:

$$(3|5)^2 = 3^2|2 \times 3 \times 5|5^2,$$

where $x|y|z$ represents $100x + 10y + z$.

This method is not fast. To square $x|y$ requires squaring x , squaring y , and multiplying x and y (plus a few additions, but those are quick). But isn't that easy, since $x = 3$ and $y = 5$? In this case, it is easy. However, I

want the squaring algorithm to work for giant integers, where x and y are themselves giant integers. So, the algorithm will be used recursively.

Now I'll estimate S_n , the time required to square an n -digit number. The algorithm requires two squarings of $n/2$ -digit numbers. It also requires multiplying two $n/2$ -digit numbers (x and y). Then

$$S_n = 2S_{n/2} + M_{n/2},$$

where $M_{n/2}$ is the time required to multiply two $n/2$ -digit numbers.

To estimate S_n , I need to estimate M_n . Using a similar algorithm as for squaring, multiplying two n -digit numbers involves four multiplications of $n/2$ -digit numbers. So

$$M_n = 4M_{n/2}.$$

This recurrence has the solution $M_n \propto n^2$. Call the constant of proportionality A , so $M_n = An^2$.

Then the recurrence for S_n , the time to square an n -digit number, becomes

$$S_n = 2S_{n/2} + \frac{A}{4}n^2.$$

To solve this recurrence, I guess that squaring is not tremendously faster than multiplying. So S_n is not going to be proportional to n or even $n \log n$, and is likely to be proportional to n^2 . This guess goes by the fancy name of an Ansatz.

Let B be the constant of proportionality: $S_n = Bn^2$. Then the recurrence for the squaring time becomes:

$$Bn^2 = 2\frac{B}{4}n^2 + \frac{A}{4}n^2.$$

The common n^2 factors divide out, leaving behind

$$B = \frac{B}{2} + \frac{A}{4},$$

whose solution is $B = A/2$. Since this equation is not nonsense, the guess is very likely to be valid. The result is that squaring using the common method is a quadratic operation (as is multiplying).

A slight variation in the common method makes it significantly faster. The problem with the common method is that it uses multiplication, which is quadratic (at least using a similar multiplication algorithm), and the slow method of multiplication contaminates the squaring algorithm. If only

there were a way to avoid multiplying! And there is! To square $x|y$, compute $(x|y)^2$ as follows:

$$(x|y)^2 = x^2|x^2 + y^2 - (x - y)^2|y^2.$$

This new method is significantly faster, as I show with the next estimate. Let S'_n be the time to square an n -digit number using this new method. It requires squaring three $n/2$ -digit numbers: x , y , and $x - y$. So

$$S'_n = 3S'_{n/2}.$$

This recurrence has the solution

$$S_n \propto n^{\log_2 3} \approx n^{1.58}$$

The exponent is roughly 1.58 instead of 2. This small decrease has a large effect when n is large. For example, when multiplying billion-digit numbers, the ratio of n^2 to $n^{\log_2 3}$ is roughly 5000.

Why would anyone multiply billion-digit numbers? One answer is to compute π to a billion digits. Why would anyone do that? Computing π to a huge number of digits, and comparing the result with the calculations of other supercomputers, is one way to check the numerical hardware in a new supercomputer.

I haven't told the whole story. The fast algorithm, known as the Karatsuba algorithm after its inventor [18], is not used for absurdly huge numbers. For large enough n , an algorithm using fast Fourier transforms is still faster. The so-called Schönhage–Strassen algorithm [32] requires a time proportional to $n \log n \log \log n$. High-quality libraries for large-number multiplication use a combination of regular multiplication, Karatsuba, and Schönhage–Strassen, selecting the algorithm according to the size of the integer.

3.6 Spacetime

An abstraction can be so useful as to be unbelievable. An example is the concept of spacetime, introduced in a famous lecture on relativity given by the mathematician Hermann Minkowski [22]. Minkowski boldly announced (translation from [37]):

From this hour on, space by itself and time by itself are to sink fully into the shadows and only a kind of union of the two should yet preserve autonomy.

That union – spacetime – is the fundamental abstraction of our physical theories. As an example, Feynman’s Nobel Prize is for what he named a ‘spacetime view of quantum electrodynamics’ [11].

Yet at its birth spacetime was so surprising and novel that even Einstein, who created the theory of relativity, called spacetime ‘superfluous learnedness’, saying of it: ‘Since the mathematicians have attacked the relativity theory, I myself no longer understand it any more.’ However, by 1916 Einstein had made spacetime the fundamental actor in his theory of gravity (general relativity). Today the abstraction of spacetime is almost an axiom of physics: A proposed theory that is not relativistically invariant – i.e. that is not based on spacetime – is today rejected out of hand. Let’s look at this idea, so useful and so unbelievable.

Spacetime is the union of space and time, separate concepts from Newtonian physics. The theory of relativity mixed space and time, but, as presented by Einstein, did so in a way that was not obvious. Imagine observer A at rest and observer B moving to the right with speed v . At time 0 the two observers coincide. Now imagine an event, such as a firecracker exploding. Observer A says it happened at time t_A and location x_A . What does observer B say? That it happened at the same time, $t_B = t_A$, and at position $x_B = x_A - vt_A$. So the transformation law between the two coordinate systems is:

$$\begin{aligned} t_B &= t_A; \\ x_B &= x_A - vt_A. \end{aligned}$$

This Galilean transformation law was revised by Einstein and Lorentz. The Lorentz transformation law is

$$\begin{aligned} x_B &= \frac{x_A - vt_A}{\sqrt{1 - v^2/c^2}}; \\ t_B &= \frac{t_A - vx_A/c^2}{\sqrt{1 - v^2/c^2}}. \end{aligned}$$

As a check, the limiting case $v \ll c$ makes the Lorentz transformation into the Galilean transformation. So in ordinary affairs we do not notice the difference. But we like our physical laws to be valid for a wide range of phenomena.

Let’s make abstractions bit by bit. First, define velocity as ‘relative to the speed of light c ’. In other words, $c = 1$ in this system. The equations become

$$x_B = \frac{x_A - vt_A}{\sqrt{1 - v^2}};$$

$$t_B = \frac{t_A - vx_A}{\sqrt{1 - v^2}}.$$

Second, name the magic factor $1/\sqrt{1 - v^2}$. The standard name is γ . So the equations become

$$x_B = \gamma(x_A - vt_A);$$

$$t_B = \gamma(t_A - vx_A).$$

The third abstraction, halfway to spacetime, is to rewrite the transformations in matrix form:

$$\begin{pmatrix} t_B \\ x_B \end{pmatrix} = \gamma \begin{pmatrix} 1 & -v \\ -v & 1 \end{pmatrix} \begin{pmatrix} t_A \\ x_A \end{pmatrix}.$$

The next abstraction is to focus on the transformation itself. Let's give it a name:

$$L = \gamma \begin{pmatrix} 1 & -v \\ -v & 1 \end{pmatrix}$$

Its determinant $|L|$ is 1, so it behaves like a rotation. Except that rotation matrices have the form

$$R_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.$$

As in L , the diagonal elements are equal. But unlike L , a rotation matrix has the off-diagonal elements as negatives of each other.

But, if time is replaced by imaginary time, then the Lorentz transformation becomes

$$\begin{pmatrix} it_B \\ x_B \end{pmatrix} = \gamma \begin{pmatrix} 1 & -iv \\ -v/i & 1 \end{pmatrix} \begin{pmatrix} t_A \\ x_A \end{pmatrix}.$$

The transformation matrix itself is then

$$\gamma \begin{pmatrix} 1 & -iv \\ iv & 1 \end{pmatrix}$$

The determinant of the transformation is still 1, and now it has the form of a rotation matrix. [The phrase 'has the form of' means an abstraction is about to follow!] The rotation is through the imaginary angle $\arctan iv$. Then the transformation is a pure rotation.

So, the complex mess of the Lorentz transformation turns into a familiar object, a rotation. The current fashion is to avoid the imaginary time axis, and not to think of the Lorentz transformation as a Euclidean rotation. Instead, it is a transformation of Minkowski space, which is a lot like Euclidean space but has a slightly strange transformation law.

The essential characteristic of a transformation is: What does it leave unchanged? The answer to that question is the topic of ???. The point of this section was to enable that question, for without the idea of spacetime we could not even ask the question.

3.7 Example: Diagrams

A powerful kind of abstraction is abstraction using diagrams. Examples include the tree analysis of recursion in [Section 3.5](#) and of divide-and-conquer reasoning in [Section 2.3](#).

A famous kind of diagram is the Feynman diagram. It is a kind of space-time diagram, and perhaps thousands of papers in physics use them; modern physics depends on that kind of diagram [\[17\]](#).

Diagrams have the general virtues of all abstractions: better thinking by naming and chunking complex ideas. Diagrams also benefit from a contingent fact of the human mind, that our vision hardware is so much more powerful than our symbolic-analysis hardware. There are evolutionary grounds to explain the discrepancy. Symbolic analysis requires processing sequences. Our capacity for sequential analysis took off with the advent of language perhaps 100,000 years ago. Visual processing, however, has developed for millions of years in the primate line alone, and even longer among vertebrates generally. So our brain has developed much more in that direction. That is why visual learning often requires only one presentation. Once you see the dog in Richard Gregory's famous obscured picture of a dalmation [\[27\]](#), you see it again very easily even ten years later (John Allman, personal communication). If only we could say that about learning symbolic information. I now know the Navier–Stokes equation,

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{v},$$

but learning it required many, many presentations!

As another example of the discrepancy, even simple examples (for us!), such as spotting one's grandmother in a photograph of a large family gathering, are beyond the capabilities of the most powerful computers. 'I see', in colloquial English usage, means 'I understand'.

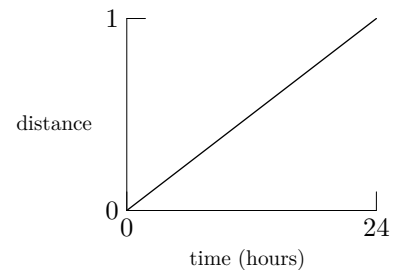
If you want to think, make abstractions. If you want to think well, make visual abstractions – make diagrams.

3.7.1 Phase space

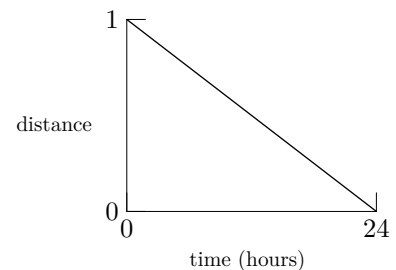
Here is a problem that is hard to solve without the abstraction of a diagram. You hike a path up a mountain over a 24-hour period, resting along the way as you need. You sleep for 24 hours at the top. Then you walk down the same path over the next 24 hours. Were you at any point on the path at the same time of day on the way up and the way down? Or can you walk up and down on a carefully crafted schedule so that there is no such point?

To solve this problem, make abstractions by deciding on the details that you don't care about. For example, the day of the month, the year, or the age of the hiker are irrelevant to the problem. A particular walking and resting schedule can be abstracted to a function of t , the time of day, that gives the the distance along the path. Let $u(t)$ be the schedule going up, and $d(t)$ be the schedule going down. In this representation, the question is: Must $u(t) = d(t)$ for some value of t ? Or can you choose $u(t)$ and $d(t)$ to avoid the equality for all values of t ?

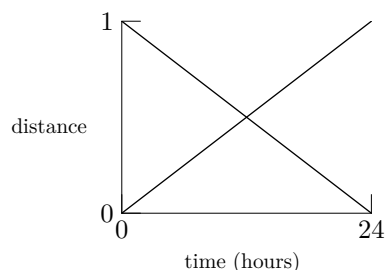
In that representation, the question is perhaps more clearly stated. But it is not much easier to answer. A diagrammatic representation makes the answer more obvious. Here is a diagram illustrating an upward schedule. Distance is measured from the bottom of the mountain (0) to the top of the mountain (1). On this schedule, you walked fast (the initial slope), rested (the flat part), and then walked to the top.



Here is a diagram illustrating a downward schedule. On this schedule, you rested (initial flat line), walked fast, then walked slowly to the bottom.



And this diagram shows the upward and downward schedules on the same diagram. And something interesting happens: The paths intersect! The intersection point gives the time and location where the upward and downward schedules landed on the same point at the same time of day (but on different days, of course).



This pattern is general. No matter what the schedules are, the upward and downward paths cross somewhere. So the answer to the question is yes, and the answer would be very hard to reach without a diagram.

This diagram has two names: a phase-space diagram or a spacetime diagram. Both types are useful in science and engineering. Spacetime diagrams, used in Einstein's theory of relativity, are the subject of the wonderful textbook [34]. They are the essential ingredient in a famous representation: Richard Feynman's diagrams for calculations in the theory of quantum electrodynamics (how radiation interacts with matter). Those diagrams are discussed in [35] and [17].

3.8 More with abstraction

The main ideas in this chapter:

For more on the value of diagrams, see [33] and [23].

Problem 3.2 Spacetime diagrams

Learn about spacetime diagrams. My favorite source is *Spacetime Physics* [34].

Problem 3.3 Word processors

Compare WYSIWIG (what you see is what you get) word processors such as WordPerfect or Microsoft Word with document formatting systems such as $\text{\TeX}$ or $\text{\ConTeXt}$ (used to typeset this book).

Problem 3.4 Longest left-handed word

What is the longest word in the dictionary that can be typed with only the left hand (on a qwerty keyboard)?

Part 2

Lossless compression

4. Symmetry and conservation	64
5. Proportional reasoning	81
6. Dimensions	96

The first part discussed methods for organizing and therefore for managing complexity. The remaining two parts discuss how to *discard complexity*. The discarded complexity can be actual complexity (**Part 3**) or it can be only apparent complexity – whereupon discarding it does not discard information. Such lossless compression is the subject of this part.

The three methods are symmetry and conservation, proportional reasoning, and dimensional analysis. Proportional reasoning and dimensional analysis are, additionally, examples of symmetry reasoning. Therefore, the next chapter introduces symmetry and conservation reasoning.

Chapter 4

Symmetry and conservation

When symmetry can be applied to a problem, it often greatly simplifies the problem – and at no cost in accuracy. A classic example is a story about the young Carl Friedrich Gauss. The story is perhaps an urban legend, but it is so instructive that it ought to be true.

When Gauss was 3 years old, the story goes, his schoolteacher wanted to occupy the young students for a good while. So he asked them to compute

$$S = 1 + 2 + 3 + \cdots + 100.$$

To the teacher's surprise, Gauss returned in just a few minutes claiming that the sum is 5050. Was he right? If so, how did he do it so quickly?

Gauss noticed that the sum remains fixed if the terms are added backwards, from last to first. In other words,

$$S' = 100 + 99 + 98 + \cdots + 1$$

equals S . Then add these two ways to compute S :

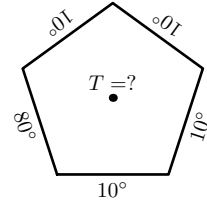
$$\begin{array}{r} S = 1 + 2 + 3 + \cdots + 100 \\ +S = 100 + 99 + 98 + \cdots + 1 \\ \hline 2S = 101 + 101 + \cdots + 101. \end{array}$$

In this form, $2S$ is easy to compute because it is 100 copies of 101. So $2S = 100 \times 101$ and $S = 50 \times 101 = 5050$.

Gauss found a symmetry, and it tremendously simplified the problem. In order to extract a general pattern to reuse in other areas, let's try symmetry in diverse examples.

4.1 Heat flow

Imagine a metal sheet, perhaps aluminum foil, cut in the shape of a regular pentagon. Attach heat sources and sinks to the edge that hold the five edges at the temperatures marked on the figure. After enough time passes, the temperature distribution in the pentagon stops changing ('comes to equilibrium'). What then is the temperature at the center of the pentagon?



A brute-force analytic solution is difficult. Heat flow is described by the following second-order partial differential equation:

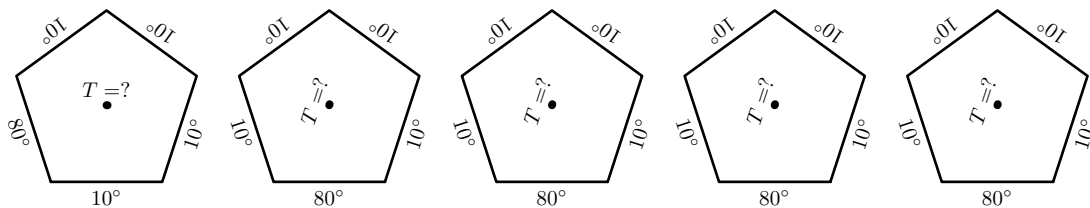
$$\kappa \nabla^2 T = \frac{\partial T}{\partial t},$$

where T is the temperature as a function of position and time, and κ is a constant known as the thermal diffusivity. Waiting makes time derivatives approach zero (everything eventually settles down), so in our problem the right side is zero. Therefore, the equation simplifies to

$$\kappa \nabla^2 T = 0.$$

Alas, even this simpler time-independent equation has simple solutions only for a few simple boundaries. A pentagon, even a regular pentagon, is not among those boundaries.

Symmetry, however, makes the solution flow. Rotating the pentagon about its center does not change the temperature at the center. Nature, in the person of the heat equation, does not care in what direction our coordinate system points. Mathematically stated, the laplacian operator ∇^2 is rotation invariant. So these five orientations of the pentagon behave identically:



Now stack these sheets (mentally), adding the temperatures that lie on top of each other to make the temperature profile of a new metal supersheet. On this new sheet, each edge has temperature

$$T_{\text{edge}} = 80^\circ + 10^\circ + 10^\circ + 10^\circ + 10^\circ = 120^\circ.$$

To solve this resulting temperature distribution, there is no need to solve the heat equation. Since all the edges are held at 120° , the temperature throughout the sheet is 120° .

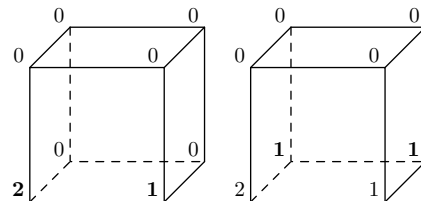
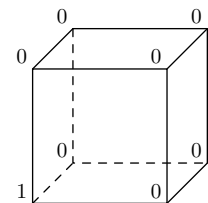
That information is enough to solve the original problem. The symmetry operation is a rotation about the center of the pentagon, so the centers overlap when the plates are stacked atop one another. Since the stacked plate has a temperature of 120° throughout, and the centers of the five stacked sheets align, each center is at $T = 120^\circ/5 = 24^\circ$.

To find transferable ideas, compare the symmetry solutions to Gauss's sum and to the pentagon temperature. Both problems looked complex at first glance. Gauss's sum had many terms in it, all different. The pentagon problem seemed to require solving a difficult differential equation. Both problems contained a symmetry operation. In Gauss's sum, the symmetry operation flipping the sum around. In the pentagon problem, the symmetry operation rotated the pentagon by 72° . In both problems, the symmetry operation left an important quantity unchanged: the sum S or the temperature T_{center} . And this invariance became the key to solving the problem simply.

A moral of these two examples is: When there is change, look for what does not change. In other words, look for invariants. Alternatively, if those quantities are given (e.g. the sum S or temperature at the center), look for operations that leave them unchanged. In other words, look for symmetries.

4.2 Cube solitaire

Here is a game of solitaire that illustrates the theme of this chapter. The following cube starts in the configuration in the margin; the goal is to make all vertices be multiples of three simultaneously. The moves are all of the same form: Pick any edge and increment its two vertices by one. For example, if I pick the bottom edge of the front face, then the bottom edge of the back face, the configuration becomes the first one in this series, then the second one:

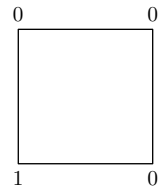


Alas, neither configuration wins the game.

Can I win the cube game? If I can win, what is a sequence of moves ends in all vertices being multiples of 3? If I cannot win, how can that negative result be proved?

Brute force – trying lots of possibilities – looks overwhelming. Each move requires choosing one of 12 edges, so there are 12^{10} sequences of ten moves. That number is an overestimate because the order of the moves does not affect the final state. I could push that line of reasoning by figuring out how many possibilities there are, and how to list and check them if the number is not too large. But that approach is specific to this problem and unlikely to generalize to other problems.

Instead of that specific approach, make the generic observation that this problem is difficult because each move offers many choices. The problem would be simpler with fewer edges: for example, if the cube were a square. Can this square be turned into one where the four vertices are multiples of 3? This problem is not the original problem, but solving it might teach me enough to solve the cube. This hope motivates the following advice: *When the going gets tough, the tough lower their standards.*



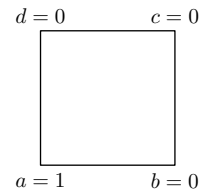
The square is easier to analyze than is the cube, but standards can be lowered still more by analyzing the one-dimensional analog, a line. Having only one edge means that there is only one move: incrementing the top and bottom vertices. The vertices start with a difference of one, and continue with that difference. So they cannot be multiples of 3 simultaneously. In symbols: $a - b = 1$. If all vertices were multiples of 3, then $a - b$ would also be a multiple of 3. Since $a - b = 1$, it is also true that



$$a - b \equiv 1 \pmod{3},$$

where the mathematical notation $x \equiv y \pmod{3}$ means that x and y have the same remainder (the same modulus) when dividing by 3. In this one-dimensional version of the game, the quantity $a - b$ is an *invariant*: It is unchanged after the only move of increasing each vertex on an edge.

Perhaps a similar invariant exists in the two-dimensional version of the game. Here is the square with variables to track the number at each vertex. The one-dimensional invariant $a - b$ is sometimes an invariant for the square. If my move uses the bottom edge, then a and b increase by 1, so $a - b$ does not change. If my move uses the top edge, then a and b are individually unchanged so $a - b$ is again unchanged. However, if my move uses the left



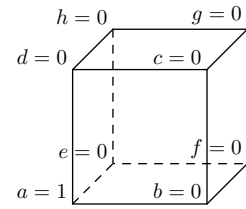
or right edge, then either a or b changes without a compensating change in the other variable. The difference $d - c$ has a similar behavior in that it is changed by some of the moves. Fortunately, even when $a - b$ and $d - c$ change, they change in the same way. A move using the left edge increments $a - b$ and $d - c$; a move using the right edge decrements $a - b$ and $d - c$. So $(a - b) - (d - c)$ is invariant! Therefore for the square,

$$a - b + c - d \equiv 1 \pmod{3},$$

so it is impossible to get all vertices to be multiples of 3.

The original, three-dimensional solitaire game is also likely to be impossible to win. The correct invariant shows this impossibility. The quantity $a - b + c - d + f - g + h - e$ generalizes the invariant for the square, and it is preserved by all 12 moves. So

$$a - b + c - d + f - g + h - e \equiv 1 \pmod{3},$$



which shows that all vertices cannot be made multiples of 3 simultaneously.

Invariants – quantities that remain unchanged – are a powerful tool for solving problems. Physics problems are also solitaire games, and invariants (conserved quantities) are essential in physics. Here is an example: In a frictionless world, design a roller-coaster track so that an unpowered roller coaster, starting from rest, rises above its starting height. Perhaps a clever combination of loops and curves could make it happen.

The rules of the physics game are that the roller coaster's position is determined by Newton's second law of motion $F = ma$, where the forces on the roller coaster are its weight and the contact force from the track. In choosing the shape of the track, you affect the contact force on the roller coaster, and thereby its acceleration, velocity, and position. There are an infinity of possible tracks, and we do not want to analyze each one to find the forces and acceleration. An invariant, energy, simplifies the analysis. No matter what tricks the track does, the kinetic plus potential energy

$$\frac{1}{2}mv^2 + mgh$$

is constant. The roller coaster starts with $v = 0$ and height h_{start} ; it can never rise above that height without violating the constancy of the energy. The invariant – the conserved quantity – solves the problem in one step, avoiding an endless analysis of an infinity of possible paths.

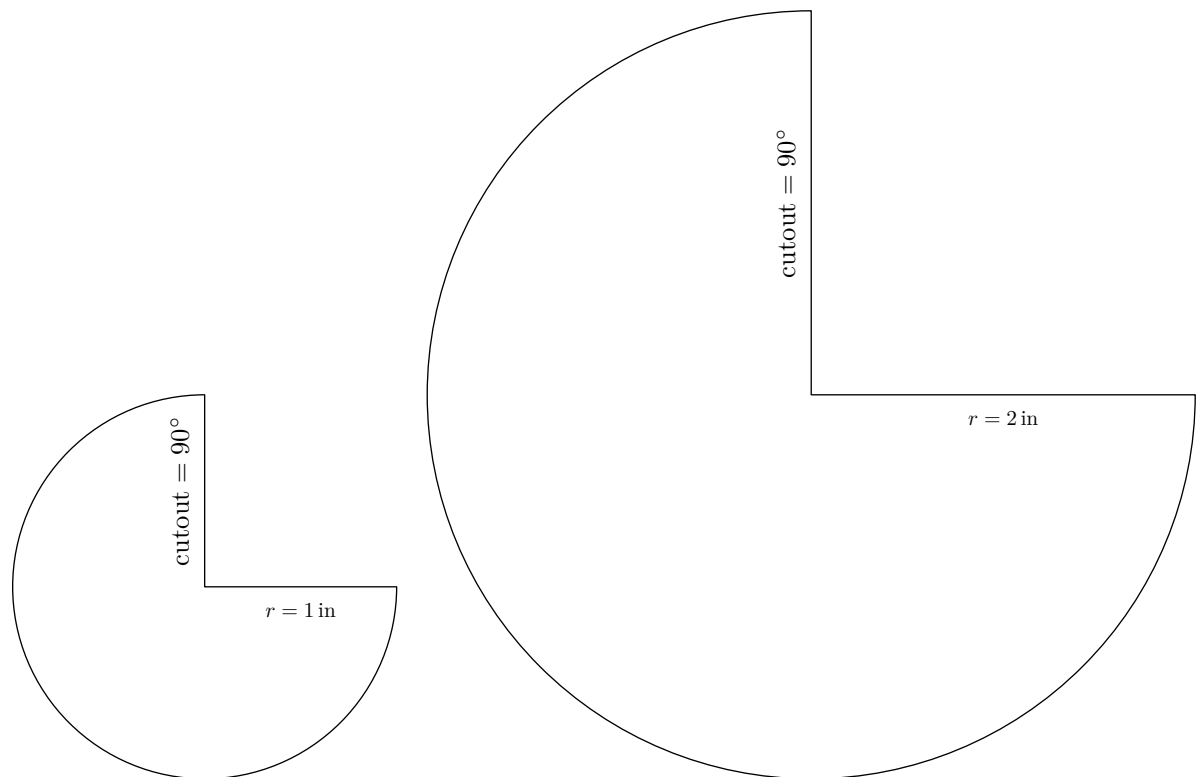
The moral of this section is: *When there is change, look for what does not change.* That quantity becomes a new abstraction (**Chapter 3**), so looking for invariants is a recipe for developing useful new abstractions.

4.3 Drag using conservation of energy

Conservation of energy helps analyze drag – one of the most difficult subjects in classical physics. To make drag concrete, try the following home experiment.

4.3.1 Home experiment using falling cones

Photocopy this page and cut out the templates, then tape their edges together to make a cone:



When you drop the small cone and the big cone, which one falls faster? In particular, what is the ratio of their fall speeds $v_{\text{big}}/v_{\text{small}}$? The large cone, having a large area, feels more drag than the small cone does. On the other hand, the large cone has a higher driving force (its weight) than the

small cone has. To decide whether the extra weight or the extra drag wins requires finding how drag depends on the parameters of the situation.

However, finding the drag force is a very complicated calculation. The full calculation requires solving the Navier–Stokes equations:

$$(\mathbf{v} \cdot \nabla) \mathbf{v} + \frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{v}.$$

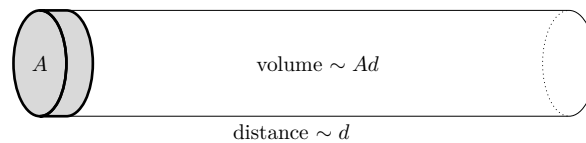
And the difficulty does not end with this set of second-order, coupled, non-linear partial-differential equations. The full description of the situation includes a fourth equation, the continuity equation:

$$\nabla \cdot \mathbf{v} = 0.$$

One imposes boundary conditions, which include the motion of the object and the requirement that no fluid enters the object – and solves for the pressure p and the velocity gradient at the surface of the object. Integrating the pressure force and the shear force gives the drag force.

In short, solving the equations analytically is difficult. I could spend hundreds of pages describing the mathematics to solve them. Even then, solutions are known only in a few circumstances, for example a sphere or a cylinder moving slowly in a viscous fluid or a sphere moving at any speed in an zero-viscosity fluid. But an inviscid fluid – what Feynman calls ‘dry water’ [12, Chapter II-40] – is particularly irrelevant to real life since viscosity is the reason for drag, so an inviscid solution predicts zero drag! Conservation of energy, supplemented with skillful lying, is a simple and quick alternative.

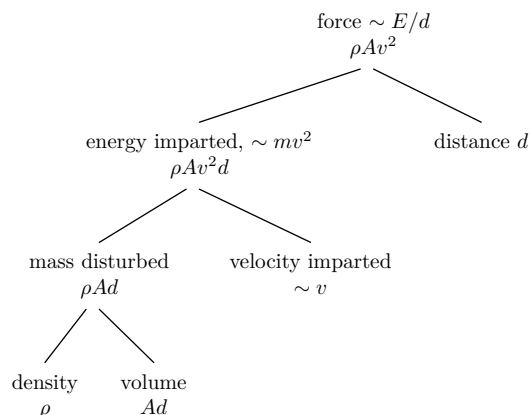
The analysis analysis imagines an object of cross-sectional area A moving through a fluid at speed v for a distance d :



The drag force is the energy consumed per distance. The energy is consumed by imparting kinetic energy to the fluid, which viscosity eventually removes from the fluid. The kinetic energy is mass times velocity squared. The mass disturbed is ρAd , where ρ is the fluid density (here, the air density). The velocity imparted to the fluid is roughly the velocity of the disturbance, which is v . So the kinetic energy imparted to the fluid is $\rho Av^2 d$, making the drag force

$$F \sim \rho A v^2.$$

The analysis has a divide-and-conquer tree:



The result that $F_{\text{drag}} \sim \rho v^2 A$ is enough to predict the result of the cone experiment. The cones reach terminal velocity quickly (see [Problem 8.6](#)), so the relevant quantity in finding the fall time is the terminal velocity. From the drag-force formula, the terminal velocity is

$$v \sim \sqrt{\frac{F_{\text{drag}}}{\rho A}}.$$

The cross-sectional areas are easy to measure with a ruler, and the ratio between the small- and large-cone terminal velocities is even easier. The experiment is set up to make the drag force easy to measure: Since the cones fall at their respective terminal velocities, the drag force equals the weight. So

$$v \sim \sqrt{\frac{W}{\rho A}}.$$

Each cone's weight is proportional to its cross-sectional area, because they are geometrically similar and made out of the same piece of paper. So the terminal velocity v is independent of the area A : so the small and large cones should fall at the same speed.

To test this prediction, I stood on a table and dropped the two cones. The fall lasted about two seconds, and they landed within 0.1 s of one another. So, the approximate conservation-of-energy analysis gains in plausibility (all the inaccuracies are hidden within the changing drag coefficient).

4.4 Cycling

This section discusses cycling as an example of how drag affects the performance of people as well as fleas. Those results will be used in the analysis of swimming, the example of the next section.

What is the world-record cycling speed? Before looking it up, predict it using armchair proportional reasoning. The first task is to define the kind of world record. Let's say that the cycling is on a level ground using a regular bicycle, although faster speeds are possible using special bicycles or going downhill.

To estimate the speed, make a model of where the energy goes. It goes into rolling resistance, into friction in the chain and gears, and into drag. At low speeds, the rolling resistance and chain friction are probably important. But the importance of drag rises rapidly with speed, so at high-enough speeds, drag is the dominant consumer of energy.

For simplicity, assume that drag is the only consumer of energy. The maximum speed happens when the power supplied by the rider equals the power consumed by drag. The problem therefore divides into two estimates: the power consumed by drag and the power that an athlete can supply.

The drag power P_{drag} is related to the drag force:

$$P_{\text{drag}} = F_{\text{drag}}v \sim \rho v^3 A.$$

It indeed rises rapidly with velocity, supporting the initial assumption that drag is the important effect at world-record speeds.

Setting $P_{\text{drag}} = P_{\text{athlete}}$ gives

$$v_{\text{max}} \sim \left(\frac{P_{\text{athlete}}}{\rho A} \right)^{1/3}$$

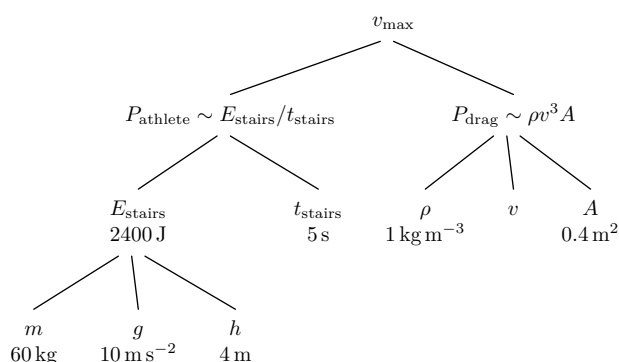
To estimate how much power an athlete can supply, I ran up one flight of stairs leading from the MIT Infinite Corridor. The Infinite Corridor, being an old building, has spacious high ceilings, so the vertical climb is perhaps $h \sim 4$ m (a typical house is 3 m per storey). Leaping up the stairs as fast as I could, I needed $t \sim 5$ s for the climb. My mass is 60 kg, so my power output was

$$\begin{aligned} P_{\text{author}} &\sim \frac{\text{potential energy supplied}}{\text{time to deliver it}} \\ &= \frac{mgh}{t} \sim \frac{60 \text{ kg} \times 10 \text{ m s}^{-2} \times 4 \text{ m}}{5 \text{ s}} \sim 500 \text{ W}. \end{aligned}$$

P_{athlete} should be higher than this peak power since most authors are not Olympic athletes. Fortunately I'd like to predict the endurance record. An Olympic athlete's long-term power might well be comparable to my peak power. So I use $P_{\text{athlete}} = 500 \text{ W}$.

The remaining item is the cyclist's cross-sectional area A . Divide the area into width and height. The width is a body width, perhaps 0.4 m . A racing cyclist crouches, so the height is maybe 1 m rather than a full 2 m . So $A \sim 0.4 \text{ m}^2$.

Here is the tree that represents this analysis:



Now combine the estimates to find the maximum speed. Putting in numbers gives

$$v_{\text{max}} \sim \left(\frac{P_{\text{athlete}}}{\rho A} \right)^{1/3} \sim \left(\frac{500 \text{ W}}{1 \text{ kg m}^{-3} \times 0.4 \text{ m}^2} \right)^{1/3}.$$

The cube root might suggest using a calculator. However, massaging the numbers simplifies the arithmetic enough to do it mentally. If only the power were 400 W or, instead, if the area were 0.5 m^2 ! Therefore, in the words of Captain Jean-Luc Picard, 'make it so'. The cube root becomes easy:

$$v_{\text{max}} \sim \left(\frac{400 \text{ W}}{1 \text{ kg m}^{-3} \times 0.4 \text{ m}^2} \right)^{1/3} \sim (1000 \text{ m}^3 \text{ s}^{-3})^{1/3} = 10 \text{ m s}^{-1}.$$

So the world record should be, if this analysis has any correct physics in it, around 10 m s^{-1} or 22 mph .

The world one-hour record – where the contestant cycles as far as possible in one hour – is 49.7 km or 30.9 mi . The estimate based on drag is reasonable!

4.5 Flight

How far can birds and planes fly? The theory of flight is difficult and involves vortices, Bernoulli's principle, streamlines, and much else. This section offers an alternative approach: use conservation estimate the energy required to generate lift, then minimize the lift and drag contributions to the energy to find the minimum-energy way to make a trip.

4.5.1 Lift

Instead of wading into the swamp of vortices, study what does not change. In this case, the vertical component of the plane's momentum does not change while it cruises at constant altitude.

Because of momentum conservation, a plane must deflect air downward. If it did not, gravity would pull the plane into the ground. By deflecting air downwards – which generates lift – the plane gets a compensating, upward recoil. Finding the necessary recoil leads to finding the energy required to produce it.

Imagine a journey of distance s . I calculate the energy to produce lift in three steps:

1. How much air is deflected downward?
2. How fast must that mass be deflected downward in order to give the plane the needed recoil?
3. How much kinetic energy is imparted to that air?

The plane is moving forward at speed v , and it deflects air over an area L^2 where L is the wingspan. Why this area L^2 , rather than the cross-sectional area, is subtle. The reason is that the wings disturb the flow over a distance comparable to their span (the longest length). So when the plane travels a distance s , it deflects a mass of air

$$m_{\text{air}} \sim \rho L^2 s.$$

The downward speed imparted to that mass must take away enough momentum to compensate for the downward momentum imparted by gravity. Traveling a distance s takes time s/v , in which time gravity imparts a downward momentum Mgs/v to the plane. Therefore

$$m_{\text{air}} v_{\text{down}} \sim \frac{Mgs}{v}$$

so

$$v_{\text{down}} \sim \frac{Mgs}{vm_{\text{air}}} \sim \frac{Mgs}{\rho v L^2 s} = \frac{Mg}{\rho v L^2}.$$

The distance s divides out, which is a good sign: The downward velocity of the air should not depend on an arbitrarily chosen distance!

The kinetic energy required to send that much air downwards is $m_{\text{air}} v_{\text{down}}^2$. That energy factors into $(m_{\text{air}} v_{\text{down}}) v_{\text{down}}$, so

$$E_{\text{lift}} \sim \underbrace{m_{\text{air}} v_{\text{down}}}_{Mgs/v} v_{\text{down}} \sim \frac{Mgs}{v} \underbrace{\frac{Mg}{\rho v L^2}}_{v_{\text{down}}} = \frac{(Mg)^2}{\rho v^2 L^2} s.$$

Check the dimensions: The numerator is a squared force since Mg is a force, and the denominator is a force, so the expression is a force times the distance s . So the result is an energy.

Interestingly, the energy to produce lift decreases with increasing speed. Here is a scaling argument to make that result plausible. Imagine doubling the speed of the plane. The fast plane makes the journey in one-half the time of the original plane. Gravity has only one-half the time to pull the plane down, so the plane needs only one-half the recoil to stay aloft. Since the same mass of air is being deflected downward but with half the total recoil (momentum), the necessary downward velocity is a factor of 2 lower for the fast plane than for the slow plane. This factor of 2 in speed lowers the energy by a factor of 4, in accordance with the v^{-2} in E_{lift} .

4.5.2 Optimization including drag

The energy required to fly includes the energy to generate lift and to fight drag. I'll add the lift and drag energies, and choose the speed that minimizes the sum.

The energy to fight drag is the drag force times the distance. The drag force is usually written as

$$F_{\text{drag}} \sim \rho v^2 A,$$

where A is the cross-sectional area. The missing dimensionless constant is $c_d/2$:

$$F_{\text{drag}} = \frac{1}{2} c_d \rho v^2 A,$$

where c_d is the drag coefficient.

However, to simplify comparing the energies required for lift and drag, I instead write the drag force as

$$F_{\text{drag}} = C\rho v^2 L^2,$$

where C is a modified drag coefficient, where the drag is measured relative to the squared wingspan rather than to the cross-sectional area. For most flying objects, the squared wingspan is much larger than the cross-sectional area, so C is much smaller than c_d .

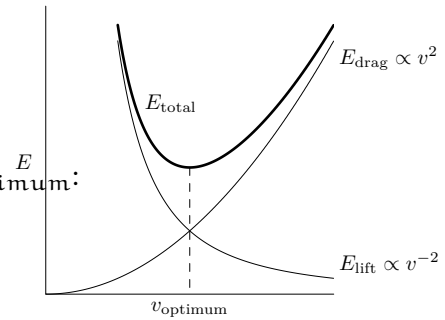
With that form for F_{drag} , the drag energy is

$$E_{\text{drag}} = C\rho v^2 L^2 s,$$

and the total energy to fly is

$$E \sim \underbrace{\frac{(Mg)^2}{\rho v^2 L^2} s}_{E_{\text{lift}}} + \underbrace{C\rho v^2 L^2 s}_{E_{\text{drag}}}.$$

A sketch of the total energy versus velocity shows interesting features. At low speeds, lift is the dominant consumer because of its v^{-2} dependence. At high speeds, drag is the dominant consumer because of its v^2 dependence. In between these extremes is an optimum speed v_{optimum} : the speed that minimizes the energy consumption for a fixed journey distance s . Going faster or slower than the optimum speed means consuming more energy. That extra consumption cannot always be avoided. A plane is designed so that its cruising speed is its minimum-energy speed. So at takeoff and landing, when its speed is much less than the minimum-energy speed, a plane requires a lot of power to stay aloft, which is one reason that the engines are so loud at takeoff and landing (another reason is probably that the engine noise reflects off the ground and back to the plane).



The constraint, or assumption, that a plane travels at the minimum-energy speed simplifies the expression for the total energy. At the minimum-energy speed, the drag and lift energies are equal. So

$$\frac{(Mg)^2}{\rho v^2 L^2} s \sim C\rho v^2 L^2 s,$$

or

$$Mg \sim C^{1/2} \rho v^2 L^2.$$

This constraint simplifies the total energy. Instead of simplifying the sum, simplify just the drag, which neglects only a factor of 2 since drag and lift are roughly equal at the minimum-energy speed. So

$$E \sim E_{\text{drag}} \sim C \rho v^2 L^2 s \sim C^{1/2} Mgs.$$

This result depends in reasonable ways upon M , g , C , and s . First, lift overcomes gravity, and gravity produces the plane's weight Mg . So Mg should show up in the energy, and the energy should, and does, increase when Mg increases. Second, a streamlined plane should use less energy than a bluff, blocky plane, so the energy should, and does, increase as the modified drag coefficient C increases. Third, since the flight is at a constant speed, the energy should be, and is, proportional to the distance traveled s .

4.5.3 Explicit computations

To get an explicit range, estimate the fuel fraction β , the energy density $\mathcal{E}$, and the drag coefficient C . For the fuel fraction I'll guess $\beta \sim 0.4$. For $\mathcal{E}$, look at the nutrition label on the back of a pack of butter. Butter is almost all fat, and one serving of 11 g provides 100 Cal (those are 'big calories'). So its energy density is 9 kcal g^{-1} . In metric units, it is $4 \cdot 10^7 \text{ J kg}^{-1}$. Including a typical engine efficiency of one-fourth gives

$$\mathcal{E} \sim 10^7 \text{ J kg}^{-1}.$$

The modified drag coefficient needs converting from easily available data. According to Boeing, a 747 has a drag coefficient of $C' \approx 0.022$, where this coefficient is measured using the wing area:

$$F_{\text{drag}} = \frac{1}{2} C' A_{\text{wing}} \rho v^2.$$

Alas, this formula is a third convention for drag coefficients, depending on whether the drag is referenced to the cross-sectional area A , wing area A_{wing} , or squared wingspan L^2 .

It is easy to convert between the definitions. Just equate the standard definition

$$F_{\text{drag}} = \frac{1}{2} C A_{\text{wing}} \rho v^2.$$

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to our definition

$$F_{\text{drag}} = CL^2 \rho v^2$$

to get

$$C = \frac{1}{2} \frac{A_{\text{wing}}}{L^2} C' = \frac{1}{2} \frac{l}{L} C',$$

since $A_{\text{wing}} = Ll$ where l is the wing width. For a 747, $l \sim 10$ m and $L \sim 60$ m, so $C \sim 1/600$.

Combine the values to find the range:

$$s \sim \frac{\beta \mathcal{E}}{C^{1/2} g} \sim \frac{0.4 \times 10^7 \text{ J kg}^{-1}}{(1/600)^{1/2} \times 10 \text{ m s}^{-2}} \sim 10^7 \text{ m} = 10^4 \text{ km}.$$

The maximum range of a 747-400 is 13,450 km, so the approximate analysis of the range is unreasonably accurate.

Problem 4.1 Integrals

Evaluate these definite integrals:

- $\int_{-10}^{10} x^3 e^{-x^2} dx$
- $\int_{-\infty}^{\infty} \frac{x^3}{1 + 7x^2 + 18x^8} dx$

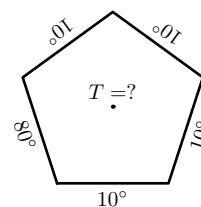
Problem 4.2 Number sum

Use symmetry to find the sum of the integers between 200 and 300 (inclusive).

Problem 4.3 Heat equation

In lecture we used symmetry to argue that the temperature at the center of the metal sheet is the average of the temperatures of the sides.

Check this result by making a simulation or, if you are bold but crazy, by finding an analytic solution of the heat equation.

**Problem 4.4 Symmetry for algebra**

Use symmetry to find $(a - b)^3$.

Problem 4.5 Symmetry for second-order systems

This problem analyzes the frequency of maximum gain for an LRC circuit or, equivalently, for a damped spring-mass system. The gain of such a system is the ratio of the input amplitude to the output amplitude as a function of frequency.

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If the output voltage is measured across the resistor, and you drive the circuit with a voltage oscillating at frequency ω , the gain is (in a suitable system of units):

$$G(\omega) = \frac{j\omega}{1 + j\omega/Q - \omega^2},$$

where $j = \sqrt{-1}$ and Q is quality factor, a dimensionless measure of the damping.

Do not worry if you do not know where that gain formula comes from. The purpose of this problem is not its origin, but rather using symmetry to maximize its magnitude.

- a. Show that the magnitude of the gain is

$$|G(\omega)| = \frac{\omega}{\sqrt{(1 - \omega^2)^2 + \omega^2/Q^2}}.$$

- b. Find a variable substitution (a symmetry operation) $\omega_{\text{new}} = f(\omega)$ that turns $|G(\omega)|$ into $|H(\omega_{\text{new}})|$ such that G and H are the same function (i.e. they have the same structure but with ω in G replaced by ω_{new} in H).
- c. Use the form of that symmetry operation to maximize $|G(\omega)|$ without using calculus.
- d. [Optional, for masochists!] Maximize $|G(\omega)|$ using calculus.

Problem 4.6 Inertia tensor

[For those who know about inertia tensors.] Here is the inertia tensor (the generalization of moment of inertia) of a particular object, calculated in a lousy coordinate system:

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 4 \\ 0 & 4 & 5 \end{pmatrix}$$

- a. Change coordinate systems to a set of principal axes. In other words, write the inertia tensor as

$$\begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$$

and give I_{xx} , I_{yy} , and I_{zz} . *Hint:* What properties of a matrix are invariant when changing coordinate systems?

- b. Give an example of an object with a similar inertia tensor. On Friday in class we'll have a demonstration.

Problem 4.7 Resistive grid

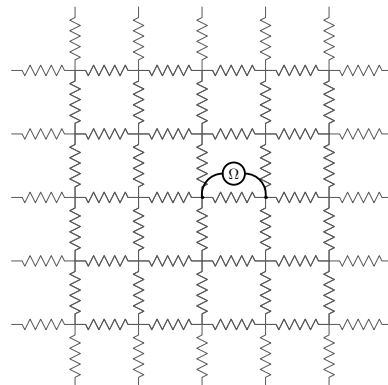
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In an infinite grid of 1-ohm resistors, what is the resistance measured across one resistor?

To measure resistance, an ohmmeter injects a current I at one terminal (for simplicity, say $I = 1\text{ A}$), removes the same current from the other terminal, and measures the resulting voltage difference V between the terminals. The resistance is $R = V/I$.

Hint: Use symmetry. But it's still a hard problem!



Chapter 5

Proportional reasoning

Symmetry wrings out excess, irrelevant complexity, and proportional reasoning in one implementation of that philosophy. If an object moves with no forces on it (or if you walk steadily), then moving for twice as long means doubling the distance traveled. Having two changing quantities contributes complexity. However, the ratio distance/time, also known as the speed, is independent of the time. It is therefore simpler than distance or time. This conclusion is perhaps the simplest example of proportional reasoning, where the proportional statement is

$$\text{distance} \propto \text{time}.$$

Using symmetry has mitigated complexity. Here the symmetry operation is 'change for how long the object move (or how long you walk)'. This operation should not change conclusions of an analysis. So, do the analysis using quantities that themselves are unchanged by this symmetry operation. One such quantity is the speed, which is why speed is such a useful quantity.

Similarly, in random walks and diffusion problems, the mean-square distance traveled is proportional to the time travelled:

$$\langle x^2 \rangle \propto t.$$

So the interesting quantity is one that does not change when t changes:

$$\text{interesting quantity} \equiv \frac{\langle x^2 \rangle}{t}.$$

This quantity is so important that it is given a name – the diffusion constant – and is tabulated in handbooks of material properties.

5.1 How maximum flight range depends on size

How does the range depend on the size of the plane? Assume that all planes are geometrically similar (have the same shape) and therefore differ only in size.

Since the energy required to fly a distance s is $E \sim C^{1/2}Mgs$, a tank of fuel gives a range of

$$s \sim \frac{E_{\text{tank}}}{C^{1/2}Mg}.$$

Let β be the fuel fraction: the fraction of the plane's mass taken up by fuel. Then $M\beta$ is the fuel mass, and $M\beta\mathcal{E}$ is the energy contained in the fuel, where $\mathcal{E}$ is the energy density (energy per mass) of the fuel. With that notation, $E_{\text{tank}} \sim M\beta\mathcal{E}$ and

$$s \sim \frac{M\beta\mathcal{E}}{C^{1/2}Mg} = \frac{\beta\mathcal{E}}{C^{1/2}g}.$$

Since all planes, at least in this analysis, have the same shape, their modified drag coefficient C is also the same. And all planes face the same gravitational field strength g . So the denominator is the same for all planes. The numerator contains β and $\mathcal{E}$. Both parameters are the same for all planes. So the numerator is the same for all planes. Therefore

$$s \propto 1.$$

All planes can fly the same distance!

Even more surprising is to apply this reasoning to migrating birds. Here is the ratio of ranges:

$$\frac{s_{\text{plane}}}{s_{\text{bird}}} \sim \frac{\beta_{\text{plane}}}{\beta_{\text{bird}}} \frac{\mathcal{E}_{\text{plane}}}{\mathcal{E}_{\text{bird}}} \left(\frac{C_{\text{plane}}}{C_{\text{bird}}} \right)^{-1/2}.$$

Take the factors in turn. First, the fuel fraction β_{plane} is perhaps 0.3 or 0.4. The fuel fraction β_{bird} is probably similar: A well-fed bird having fed all summer is perhaps 30 or 40% fat. So $\beta_{\text{plane}}/\beta_{\text{bird}} \sim 1$. Second, jet fuel energy density is similar to fat's energy density, and plane engines and animal metabolism are comparably efficient (about 25%). So $\mathcal{E}_{\text{plane}}/\mathcal{E}_{\text{bird}} \sim 1$. Finally, a bird has a similar shape to a plane – it is not a great approximation, but it has the virtue of simplicity. So $C_{\text{bird}}/C_{\text{plane}} \sim 1$.

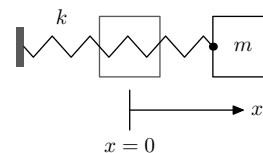
Therefore, planes and well-fed, migrating birds should have the same maximum range! Let's check. The longest known nonstop flight by an animal

is 11,570 km, made by a bar-tailed godwit from Alaska to New Zealand (tracked by satellite). The maximum range for a 747-400 is 13,450 km, only slightly longer than the godwit's range.

5.2 Period of a spring-mass system

As a first example of proportional reasoning, here is one way to explain a famous result in physics: that the period of spring-mass system is independent of the amplitude.

So imagine a mass m connected to the wall by a spring with spring constant k . If disturbed, the mass oscillates. The period of the system is the time for the mass to make a round trip through the equilibrium position.



Extend the spring by a distance x_0 ; this displacement is the amplitude. To see how it affects the period, make an approximation, which will be an example of throwing away information (the topic of ??). The approximation is to pretend that the pendulum moves with a constant speed v . Then the period is

$$T \sim \frac{\text{distance}}{\text{speed } v},$$

and the distance that the mass travels in one period is $4x_0$. Ignore the factor of 4:

$$T \sim \frac{x_0}{v}.$$

Proportional reasoning helps us estimate v by an energy argument. The initial potential energy is $PE \sim kx_0^2$ or

$$PE \propto x_0^2.$$

The maximum kinetic energy, which we use as a proxy for the typical kinetic energy, is the initial potential energy, so

$$KE_{\text{typical}} \propto x_0^2$$

as well. The typical velocity is $\sqrt{KE_{\text{typical}}}$, so

$$v_{\text{typical}} \propto x_0.$$

That result is great news because it means that the period is proportional to 1:

$$T \propto \frac{x_0}{x_0} = x_0^0.$$

In other words, the period is independent of amplitude.

5.3 Mountain heights

The next example of proportional reasoning explains why mountains cannot become too high. Assume that all mountains are cubical and made of the same material. Making that assumption discards actual complexity, the topic of ???. However, it is a useful approximation.

To see what happens if a mountain gets too large, estimate the pressure at the base of the mountain. Pressure is force divided by area, so estimate the force and the area.

The area is the easier estimate. With the approximation that all mountains are cubical and made of the same kind of rock, the only parameter distinguishing one mountain from another is its side length l . The area of the base is then l^2 .

Next estimate the force. It is proportional to the mass:

$$F \propto m.$$

In other words, F/m is independent of mass, and that independence is why the proportionality $F \propto m$ is useful. The mass is proportional to l^3 :

$$m \propto \text{volume} \sim l^3.$$

In other words, m/l^3 is independent of l ; this independence is why the proportionality $m \propto l^3$ is useful. Therefore

$$F \propto l^3.$$

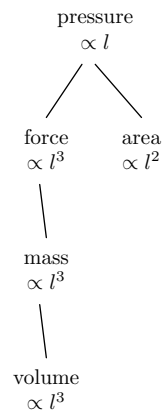
The force and area results show that the pressure is proportional to l :

$$p \sim \frac{F}{A} \propto \frac{l^3}{l^2} = l.$$

With a large-enough mountain, the pressure is larger than the maximum pressure that the rock can withstand. Then the rock flows like a liquid, and the mountain cannot grow taller.

This estimate shows only that there is a maximum height but it does not compute the maximum height. To do that next step requires estimating the strength of rock. Later in this book when we estimate the strength of materials, I revisit this example.

This estimate might look dubious also because of the assumption that mountains are cubical. Who has seen a cubical mountain? Try a reasonable alternative, that mountains are pyramidal with a square base of side l and a height l , having a 45° slope. Then the volume is $l^3/3$ instead of l^3 *but the factor of one-third does not affect the proportionality between force and length*. Because of the factor of one-third, the maximum height will be higher for a pyramidal mountain than for a cubical mountain. However, there is again a maximum size (and height) of a mountain. In general, the argument for a maximum height requires only that all mountains are similar – are scaled versions of each other – and does not depend on the shape of the mountain.



5.4 Animal jump heights

We next use proportional reasoning to understand how high animals jump, as a function of their size. Do kangaroos jump higher than fleas? We study a jump from standing (or from rest, for animals that do not stand); a running jump depends on different physics. This problem looks underspecified. The height depends on how much muscle an animal has, how efficient the muscles are, what the animal's shape is, and much else. The first subsection introduces a simple model of jumping, and the second refines the model to consider physical effects neglected in the crude approximations.

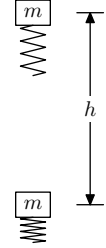
5.4.1 Simple model

We want to determine only how jump height varies with body mass. Even this problem looks difficult; the height still depends on muscle efficiency, and so on. Let's see how far we get by just plowing along, and using symbols for the unknown quantities. Maybe all the unknowns cancel.

We want an equation for the height h in the form $h \sim m^\beta$, where m is the animal's mass and β is the so-called scaling exponent.

Jumping requires energy, which must be provided by muscles. This first, simplest model equates the required energy to the energy supplied by the animal's muscles.

The required energy is the easier estimation: An animal of mass m jumping to a height h requires an energy $E_{\text{jump}} \propto m h$. Because all animals feel the same gravity, this relation does not contain the gravitational acceleration g . You could include it in the equation, but it would just carry through the equations like unused baggage on a trip.



The available energy is the harder estimation. To find it, divide and conquer. It is the product of the muscle mass and of the energy per mass (the energy density) stored in muscle.

To approximate the muscle mass, assume that a fixed fraction of an animal's mass is muscle, i.e. that this fraction is the same for all animals. If α is the fraction, then

$$m_{\text{muscle}} \sim \alpha m$$

or, as a proportionality,

$$m_{\text{muscle}} \propto m,$$

where the last step uses the assumption that all animals have the same α .

For the energy per mass, assume again that all muscle tissues are the same: that they store the same energy per mass. If this energy per mass is $\mathcal{E}$, then the available energy is

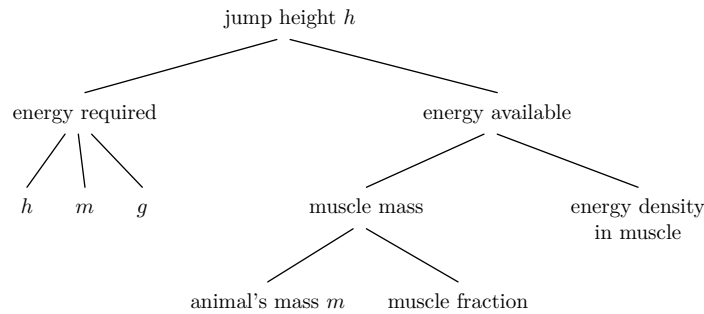
$$E_{\text{avail}} \sim \mathcal{E} m_{\text{muscle}}$$

or, as a proportionality,

$$E_{\text{avail}} \propto m_{\text{muscle}},$$

where this last step uses the assumption that all muscle has the same energy density $\mathcal{E}$.

Here is a tree that summarizes this model:



Now finish propagating toward the root. The available energy is

$$E_{\text{avail}} \propto m.$$

So an animal with three times the mass of another animal can store roughly three times the energy in its muscles, according to this simple model.

Now compare the available and required energies to find how the jump height as a function of mass. The available energy is

$$E_{\text{avail}} \propto m$$

and the required energy is

$$E_{\text{required}} \propto mh.$$

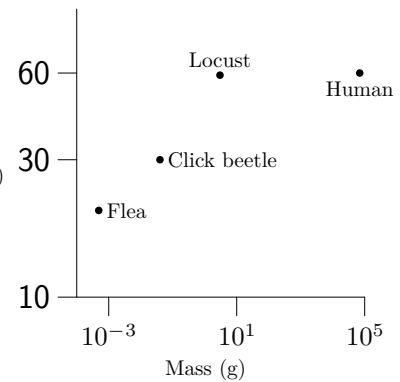
Equate these energies, which is an application of conservation of energy. Then $mh \propto m$ or

$$h \propto m^0.$$

In other words, all animals jump to the same height.

The result, that all animals jump to the same height, seems surprising. Our intuition tells us that people should be able to jump higher than locusts. The graph shows jump heights for animals of various sizes and shapes [source: *Scaling: Why Animal Size is So Important* [31, p. 178]. Here is the data:

Animal	Mass (g)	Height (cm)
Flea	$5 \cdot 10^{-4}$	20
Click beetle	$4 \cdot 10^{-2}$	30
Locust	3	59
Human	$7 \cdot 10^4$	60



The height varies almost not at all when compared to variation in mass, so our result is roughly correct! The mass varies more than eight orders of magnitude (a factor of 10^8), yet the jump height varies only by a factor of 3. The predicted scaling of constant h ($h \propto 1$) is surprisingly accurate.

5.4.2 Power limits

Power production might also limit the jump height. In the preceding analysis, energy is the limiting reagent: The jump height is determined by the energy that an animal can store in its muscles. However, even if the animal can store enough energy to reach that height, the muscles might not be able to deliver the energy rapidly enough. This section presents a simple model for the limit due to limited power generation.

Once again we'd like to find out how power P scales (varies) with the size l . Power is energy per time, so the power required to jump to a height h is

$$P \sim \frac{\text{energy required to jump to height } h}{\text{time over which the energy is delivered}}.$$

The energy required is $E \sim mgh$. The mass is $m \propto l^3$. The gravitational acceleration is independent of l . And, in the energy-limited model, the height h is independent of l . Therefore $E \propto l^3$.

The delivery time is how long the animal is in contact with the ground, because only during contact can the ground exert a force on the animal. So, the animal crouches, extends upward, and finally leaves the ground. The contact time is the time during which the animal extends upward. Time is length over speed, so

$$t_{\text{delivery}} \sim \frac{\text{extension distance}}{\text{extension speed}}.$$

The extension distance is roughly the animal's size l . The extension speed is roughly the takeoff velocity. In the energy-limited model, the takeoff velocity is the same for all animals:

$$v_{\text{takeoff}} \propto h^{1/2} \propto l^0.$$

So

$$t_{\text{delivery}} \propto l.$$

The power required is $P \propto l^3/l = l^2$.

That proportionality is for the power itself, but a more interesting scaling is for the specific power: the power per mass. It is

$$\frac{P}{m} \propto \frac{l^2}{l^3} = l^{-1}.$$

Ah, smaller animals need a higher specific power!

A model for power limits is that all muscle can generate the same maximum power density (has the same maximum specific power). So a small-enough animal cannot jump to its energy-limited height. The animal can store enough energy in its muscles, but cannot release it quickly enough.

More precisely, it cannot do so unless it finds an alternative method for releasing the energy. The click beetle, which is toward the small end in the preceding graph and data set, uses the following solution. It stores energy in its shell by bending the shell, and maintains the bending like a ratchet would (holding a structure motionless does require energy). This storage can happen slowly enough to avoid the specific-power limit, but when the beetle releases the shell and the shell snaps back to its resting position, the energy is released quickly enough for the beetle to rise to its energy-limited height.

But that height is less than the height for locusts and humans. Indeed, the largest deviations from the constant-height result happen at the low-mass end, for fleas and click beetles. To explain that discrepancy, the model needs to take into account another physical effect: drag.

5.5 Drag

5.5.1 Effect of drag on fleas jumping

The drag force

$$F \sim \rho A v^2$$

affects the jumps of small animals more than it affects the jumps of people. A comparison of the energy required for the jump with the energy consumed by drag explains why.

The energy that the animal requires to jump to a height h is mgh , if we use the gravitational potential energy at the top of the jump; or it is $\sim mv^2$, if we use the kinetic energy at takeoff. The energy consumed by drag is

$$E_{\text{drag}} \sim \underbrace{\rho v^2 A}_{F_{\text{drag}}} \times h.$$

The ratio of these energies measures the importance of drag. The ratio is

$$\frac{E_{\text{drag}}}{E_{\text{required}}} \sim \frac{\rho v^2 A h}{m v^2} = \frac{\rho A h}{m}.$$

Since A is the cross-sectional area of the animal, Ah is the volume of air that it sweeps out in the jump, and ρAh is the mass of air swept out in the jump. So the relative importance of drag has a physical interpretation as a ratio of the mass of air displaced to the mass of the animal.

To find how this ratio depends on animal size, rewrite it in terms of the animal's side length l . In terms of side length, $A \sim l^2$ and $m \propto l^3$. What about the jump height h ? The simplest analysis predicts that all animals have the same jump height, so $h \propto l^0$. Therefore the numerator ρAh is $\propto l^1$, the denominator m is $\propto l^3$, and

$$\frac{E_{\text{drag}}}{E_{\text{required}}} \propto \frac{l^2}{l^3} = l^{-1}.$$

So, small animals have a large ratio, meaning that drag affects the jumps of small animals more than it affects the jumps of large animals. The missing constant of proportionality means that we cannot say at what size an animal becomes 'small' for the purposes of drag. So the calculation so far cannot tell us whether fleas are included among the small animals.

The jump data, however, substitutes for the missing constant of proportionality. The ratio is

$$\frac{E_{\text{drag}}}{E_{\text{required}}} \sim \frac{\rho A h}{m} \sim \frac{\rho l^2 h}{\rho_{\text{animal}} l^3}.$$

It simplifies to

$$\frac{E_{\text{drag}}}{E_{\text{required}}} \sim \frac{\rho}{\rho_{\text{animal}}} \frac{h}{l}.$$

As a quick check, verify that the dimensions match. The left side is a ratio of energies, so it is dimensionless. The right side is the product of two dimensionless ratios, so it is also dimensionless. The dimensions match.

Now put in numbers. A density of air is $\rho \sim 1 \text{ kg m}^{-3}$. The density of an animal is roughly the density of water, so $\rho_{\text{animal}} \sim 10^3 \text{ kg m}^{-3}$. The typical jump height – which is where the data substitutes for the constant

of proportionality – is 60 cm or roughly 1 m. A flea's length is about 1 mm or $l \sim 10^{-3}$ m. So

$$\frac{E_{\text{drag}}}{E_{\text{required}}} \sim \frac{1 \text{ kg m}^{-3}}{10^3 \text{ kg m}^{-3}} \frac{1 \text{ m}}{10^{-3} \text{ m}} \sim 1.$$

The ratio being unity means that if a flea would jump to 60 cm, overcoming drag would require roughly as much as energy as would the jump itself in vacuum.

Drag provides a plausible explanation for why fleas do not jump as high as the typical height to which larger animals jump.

5.5.2 Swimming

The last section's analysis of cycling helps predict the world-record speed for swimming. The last section showed that

$$v_{\text{max}} \sim \left(\frac{P_{\text{athlete}}}{\rho A} \right)^{1/3}.$$

To evaluate the maximum speed for swimming, one could put in a new ρ and A directly into that formula. However, that method replicates the work of multiplying, dividing, and cube-rooting the various values.

Instead it is instructive to scale the numerical result for cycling by looking at how the maximum speed depends on the parameters of the situation. In other words, I'll use the formula for v_{max} to work out the ratio $v_{\text{swimmer}}/v_{\text{cyclist}}$, and then use that ratio along with v_{cyclist} to work out v_{swimmer} .

The speed v_{max} is

$$v_{\text{max}} \sim \left(\frac{P_{\text{athlete}}}{\rho A} \right)^{1/3}.$$

So the ratio of swimming and cycling speeds is

$$\frac{v_{\text{swimmer}}}{v_{\text{cyclist}}} \sim \left(\frac{P_{\text{swimmer}}}{P_{\text{cyclist}}} \right)^{1/3} \times \left(\frac{\rho_{\text{swimmer}}}{\rho_{\text{cyclist}}} \right)^{-1/3} \times \left(\frac{A_{\text{swimmer}}}{A_{\text{cyclist}}} \right)^{-1/3}.$$

Estimate each factor in turn. The first factor accounts for the relative athletic prowess of swimmers and cyclists. Let's assume that they generate equal amounts of power; then the first factor is unity. The second factor accounts for the differing density of the mediums in which each athlete moves. Roughly, water is 1000 times denser than air. So the second factor

contributes a factor of 0.1 to the speed ratio. If the only factors were the first two, then the swimming world record would be about 1 m s^{-1} .

Let's compare with reality. The actual world record for a 1500-m freestyle (in a 50-m pool) is 14m34.56s set in July 2001 by Grant Hackett. That speed is 1.713 m s^{-1} , significantly higher than the prediction of 1 m s^{-1} .

The third factor comes to the rescue by accounting for the relative profile of a cyclist and a swimmer. A swimmer and a cyclist probably have the same width, but the swimmer's height (depth in the water) is perhaps one-sixth that of a crouched cyclist. So the third factor contributes $6^{1/3}$ to the predicted speed, making it 1.8 m s^{-1} .

This prediction is close to the actual record, closer to reality than one might expect given the approximations in the physics, the values, and the arithmetic. However, the accuracy is a result of the form of the estimate, that the maximum speed is proportional to the cube root of the athlete's power and the inverse cube root of the cross-sectional area. Errors in either the power or area get compressed by the cube root. For example, the estimate of 500 W might easily be in error by a factor of 2 in either direction. The resulting error in the maximum speed is $2^{1/3}$ or 1.25, an error of only 25%. The cross-sectional area of a swimmer might be in error by a factor of 2 as well, and this mistake would contribute only a 25% error to the maximum speed. [With luck, the two errors would cancel!]

5.5.3 Flying

In the next example, I scale the drag formula to estimate the fuel efficiency of a jumbo jet. Rather than estimating the actual fuel consumption, which would produce a large, meaningless number, it is more instructive to estimate the relative fuel efficiency of a plane and a car.

Assume that jet fuel goes mostly to fighting drag. This assumption is not quite right, so at the end I'll discuss it and other troubles in the analysis. The next step is to assume that the drag force for a plane is given by the same formula as for a car:

$$F_{\text{drag}} \sim \rho v^2 A.$$

Then the ratio of energy consumed in travelling a distance d is

$$\frac{E_{\text{plane}}}{E_{\text{car}}} \sim \frac{\rho_{\text{up-high}}}{\rho_{\text{low}}} \times \left(\frac{v_{\text{plane}}}{v_{\text{car}}} \right)^2 \times \frac{A_{\text{plane}}}{A_{\text{car}}} \times \frac{d}{d}.$$

Estimate each factor in turn. The first factor accounts for the lower air density at a plane's cruising altitude. At 10 km, the density is roughly one-third of the sea-level density, so the first factor contributes $1/3$. The second factor accounts for the faster speed of a plane. Perhaps $v_{\text{plane}} \sim 600$ mph and $v_{\text{car}} \sim 60$ mph, so the second factor contributes a factor of 100. The third factor accounts for the greater cross-sectional area of the plane. As a reasonable estimate

$$A_{\text{plane}} \sim 6 \text{ m} \times 6 \text{ m} = 36 \text{ m}^2,$$

whereas

$$A_{\text{car}} \sim 2 \text{ m} \times 1.5 \text{ m} = 3 \text{ m}^2,$$

so the third factor contributes a factor of 12. The fourth factor contributes unity, since we are analyzing the plane and car making the same trip (New York to Los Angeles, say).

The result of the four factors is

$$\frac{E_{\text{plane}}}{E_{\text{car}}} \sim \frac{1}{3} \times 100 \times 12 \sim 400.$$

A plane looks incredibly inefficient. But I neglected the number of people. A jumbo jet takes carries 400 people; a typical car, at least in California, carries one person. So the plane and car come out equal!

This analysis leaves out many effects. First, jet fuel is used to generate lift as well as to fight drag. However, as a later analysis will show, the energy consumed in generating lift is comparable to the energy consumed in fighting drag. Second, a plane is more streamlined than a car. Therefore the missing constant in the drag force $F_{\text{drag}} \sim \rho v^2 A$ is smaller for a plane than for a car. our crude analysis of drag has not included this effect. Fortunately this error compensates, or perhaps overcompensates, for the error in neglecting lift.

5.6 Analysis of algorithms

Proportional reasoning is the basis of an entire subject of the analysis of algorithms, a core part of computer science. How fast does an algorithm run? How much space does it require? A proportional-reasoning analysis helps you decide which algorithms to use. This section discusses these decisions using the problem of how to square very large numbers.

Squaring numbers is a special case of multiplication, but the algebra is simpler for squaring than for multiplying since having only one number as the input means there are fewer variables in the analysis.

Here is a divide-and-conquer version of the standard school multiplication algorithm.

More on proportional reasoning

Summary of the chapter:

Further reading:

Problem 5.1 Raindrop speed

- How does a raindrop's terminal velocity v depend on the raindrop's radius r ?
- Estimate the terminal speed for a typical raindrop.
- How could you check your estimate in part (b)?

Problem 5.2 Mountains

Look up the height of the tallest mountain on earth, Mars, and Venus, and explain any pattern in the three values.

Problem 5.3 Highway vs city driving

In lecture we derived a measure of how important drag is for a car moving at speed v for a distance d :

$$\frac{E_{\text{drag}}}{E_{\text{kinetic}}} \sim \frac{\rho v^2 A d}{m_{\text{car}} v^2}.$$

- Show that the ratio is equivalent to the ratio

$$\frac{\text{mass of the air displaced}}{\text{mass of the car}}$$

and to the ratio

$$\frac{\rho_{\text{air}}}{\rho_{\text{car}}} \times \frac{d}{l_{\text{car}}},$$

where ρ_{car} is the density of the car (i.e. its mass divided by its volume) and l_{car} is the length of the car.

- Make estimates for a typical car and find the distance d at which the ratio becomes significant (say, roughly 1). How does the distance compare with the distance between exits on the highway and between stop signs or stoplights on city streets?

Problem 5.4 Gravity on the moon

In this problem you use a scaling argument to estimate the strength of gravity on the surface of the moon.

- Assume that a planet is a uniform sphere. What is the proportionality between the gravitational acceleration g at the surface of a planet and the planet's radius R and density ρ ?
- Write the ratio $g_{\text{moon}}/g_{\text{earth}}$ as a product of dimensionless factors as in the analysis of the fuel efficiency of planes.
- Estimate those factors and estimate the ratio $g_{\text{moon}}/g_{\text{earth}}$, then estimate g_{moon} . [Hint: To estimate the radius of the moon, whose angular size you can estimate by looking at it, you might find it useful to know that the moon is $4 \cdot 10^8$ m distant from the earth.]
- Look up g_{moon} and compare the value to your estimate, venturing an explanation for any discrepancy.

Problem 5.5 Checking plane fuel-efficiency calculation

This problem offers two more methods to estimate the fuel efficiency of a plane.

- Use the cost of a plane ticket to estimate the fuel efficiency of a 747, in passenger-miles per gallon.
- According to Wikipedia, a 747-400 can hold up to $2 \cdot 10^5 \ell$ of fuel for a maximum range of $1.3 \cdot 10^4$ km. Use that information to estimate the fuel efficiency of the 747, in passenger-miles per gallon.

How do these values compare with the rough result from lecture, that the fuel efficiency is comparable to the fuel efficiency of a car?

Chapter 6

Dimensions

6.1 Power of multinationals

The first example shows what happens when people take no notice of dimensions.

Critics of globalization often make this argument:

In Nigeria, a relatively economically strong country, the GDP [gross domestic product] is \$99 billion. The net worth of Exxon is \$119 billion. 'When multinationals have a net worth higher than the GDP of the country in which they operate, what kind of power relationship are we talking about?' asks Laura Morosini. [Source: 'Impunity for Multinationals', *ATTAC*, 11 Sept 2002, [[url:nigeria-argument](#)], retrieved 11 Sept 2006]

Before reading further, try to find the most egregious fault in the comparison between Exxon and Nigeria. It's a competitive field, but one fault stands out.

The comparison between Exxon and Nigeria has many problems. First, the comparison exaggerates Exxon's power by using its worldwide assets (net worth) rather than its assets only in Nigeria. On the other hand, Exxon can use its full international power when negotiating with Nigeria, so perhaps the worldwide assets are a fair basis for comparison.

A more serious, and less debatable, problem is the comparison with GDP, or gross domestic product. To see the problem, look at the ingredients in how GDP is usually measured: as dollars per year. The \$99 billion for Nigeria's GDP is shorthand for \$99 billion per year. A year is an astronomical time, and its use in an economic measurement is arbitrary. Economic flows, which are a social phenomenon, should not care about how long the earth requires to travel around the sun. Suppose instead that the decade was the

chosen unit of time in measuring the GDP. Then Nigeria's GDP would be roughly \$1 trillion per decade (assuming that the \$99 billion per year value held steady) and would be reported as \$1 trillion. Now Nigeria towers over the puny Exxon whose assets are a mere one-tenth of this figure.

To produce the opposite conclusion, just measure GDP in units of dollars per week: Nigeria's GDP becomes \$2 billion per week. Now puny Nigeria stands helpless before the might of Exxon, 50-fold larger than Nigeria. Either conclusion about the relative powers can be produced merely by changing the units. This arbitrariness indicates that the comparison is bogus.

The flaw in the comparison is the theme of this chapter. Assets, or net worth, are an amount of money – money is its dimensions – and are typically measured in units of dollars. GDP is defined as the total goods and services sold in one year. It is a rate and has dimensions of money per time; its typical units are dollars per year. Comparing assets to GDP means comparing money to money per time. *Because the dimensions of these two quantities are not the same, the comparison is nonsense!* A similarly flawed comparison is to compare length per time (speed) with length. Listen how ridiculous it sounds: 'I walk 1.5 meters per second, much smaller than the Empire State building in New York, which is 300 meters high.' To produce the opposite conclusion, measure time in hours: 'I walk 5000 meters per hour, much larger than the Empire State building at only 300 meters.' Nonsense all around!

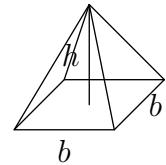
This example illustrates several ideas:

- *Dimensions versus units.* Dimensions are general and generic, such as money per time or length per time. Units are the instantiation of dimensions in a system of measurement. The most complete system of measurement is the System International (SI), where the unit of mass is the kilogram, the unit of time the second, and the unit of length the meter. Other examples of units are dollars per year or kilometers per year.
- *Necessary condition for a valid comparison.* In a valid comparison, the dimensions of the compared objects be identical. Do not compare apples to oranges (except in questions of taste, like 'I prefer apples to oranges.')
- *Rubbish abounds.* There's lots of rubbish out there, so keep your eyes open for it!
- *Bad argument, fine conclusion.* I agree with the conclusion of the article, that large oil companies exert massive power over poor countries.

However, as a physicist I am embarrassed by the reasoning. This example teaches me a valuable lesson about theorems and proofs: judge the proof not just the theorem. Even if you disagree with the conclusion, remember the general lesson that a correct conclusion does not validate a dubious argument.

6.2 Pyramid volume

The last example showed the value of dimensions in economics. The next example shows that dimensions are also useful in mathematics. What is the volume of this square-based pyramid? Here are several choices:



1. $\frac{1}{3}bh$
2. $b^3 + h^2$
3. b^4/h
4. bh^2

Let's take the choices in turn. The first choice, $bh/3$, has dimensions of area rather than volume. So it cannot be right. The second choice, $b^3 + h^2$, begins with a volume in the b^3 term but falls apart with the h^2 , which has dimensions of area. Since it adds an area to a volume – the crime of dimension mixing – it cannot be right. The third choice, b^4/h , has dimensions of volume, so it might be correct. It even increases as b increases, which is a good sign. However, the volume should increase as h increases – a proportional-reasoning argument – whereas this choice indicates that the volume decreases as h increases! So it cannot be right.

The final choice, bh^2 , has correct dimensions and increases as h or b increases. Does it increase by the right amounts? Imagine drilling into the pyramid from the top and dividing it into thin cores or volume elements. If the height of the pyramid doubles, then each vertical volume element doubles in volume; so the volume of the pyramid should double. In symbols, $V \propto h$. But bh^2 quadruples when h doubles, so that choice cannot be right.

The requirement that $V \propto h$ together with the requirement that V have dimensions of length cubed means that the missing item in $V \propto h$ is an area. The only way to make an area from b is to make b^2 perhaps times a dimensionless constant. So

$$V \sim hb^2.$$

The missing dimensionless constant is hidden in the twiddle $\sim$ sign. Alternatively, the ratio V/hb^2 is dimensionless.

This method of deducing the volume requires remembering hardly any arbitrary data. It requires these ingredients:

1. Using vertical volume elements to find out that $V \propto h$.
2. Using dimensions along with $V \propto h$ to show that $V \sim hb^2$.
3. Remembering the correct dimensionless constant.

The first two steps are logic and do not require arbitrary data. Instead they use reasoning methods that you use elsewhere (so there's no marginal cost to remember them). The third step requires seemingly arbitrary data. However, in [Chapter 7](#) on special cases, I'll show you how to determine the constant elegantly without even needing an integral.

Then the volume requires no memory. Arbitrary data is, by definition, impossible to compress. Dimensions, and more generally our techniques for handling complexity, are a form of data compression or entropy reduction [21]. One way to look at learning is as data compression. So dimensions, and our other techniques, enhance learning.

There's an old saying: Tell the truth; there's less to remember. The similar moral here is: *Use dimensions (and proportional reasoning); there's less to remember!*

6.3 Dimensionless groups

Dimensionless ratios are useful. For example, in the oil example, the ratio of the two quantities has dimensions; in that case, the dimensions of the ratio are time (or one over time). If the authors of the article had used a dimensionless ratio, they might have made a valid comparison.

This section explains why dimensionless ratios are the only quantities that you need to think about; in other words, that there is no need to think about quantities with dimensions.

To see why, take a concrete example: computing the energy E to produce lift as a function of distance traveled s , plane speed v , air density ρ , wingspan L , plane mass m , and strength of gravity g . Any true statement about these variables looks like

$$\triangle_{\text{mess}} + \square_{\text{mess}} = \bigcirc_{\text{mess}},$$

where the various messes mean 'a horrible combination of E , s , v , ρ , L , and m .

As horrible as that true statement is, it permits the following rewriting: Divide each term by the first one (the triangle). Then

$$\frac{\triangle_{\text{mess}}}{\triangle_{\text{mess}}} + \frac{\square_{\text{mess}}}{\triangle_{\text{mess}}} = \frac{\bigcirc_{\text{mess}}}{\triangle_{\text{mess}}},$$

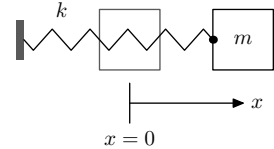
The first ratio is 1, which has no dimensions. Without knowing the individual messes, we don't know the second ratio; but it has no dimensions because it is being added to the first ratio. Similarly, the third ratio, which is on the right side, also has no dimensions.

So the rewritten expression is dimensionless. Nothing in the rewriting depended on the particular form of the true statement, except that each term has the same dimensions.

Therefore, *any true statement can be rewritten in dimensionless form.*

Dimensionless forms are made from dimensionless ratios, so all you need are dimensionless ratios, and you can do all your thinking with them. Here is a familiar example to show how this change simplifies your thinking. This example uses familiar physics so that you can concentrate on the new idea of dimensionless ratios.

The problem is to find the period of an oscillating spring-mass system given an initial displacement x_0 , then allowed to oscillate freely. [Section 5.2](#) gave a proportional-reasoning analysis of this system. The relevant variables that determine the period T are mass m , spring constant k , and amplitude x_0 . Those three variables completely describe the system, so any true statement about period needs only those variables.



Since any true statement can be written in dimensionless form, the next step is to find all dimensionless forms that can be constructed from T , m , k , and x_0 . A table of dimensions is helpful. The only tricky entry is the dimensions of a spring constant. Since the force from the spring is $F = kx$, where x is the displacement, the dimensions of a spring constant are the dimensions of force divided by the dimensions of x . It is convenient to have a notation for the concept of ‘the dimensions of’. In that notation,

Var	Dim	What
T	T	period
m	M	mass
k	MT^{-2}	spring constant
x_0	L	amplitude

$$[k] = \frac{[F]}{[x]},$$

where $[quantity]$ means the dimensions of the quantity. Since $[F] = MLT^{-2}$ and $[x] = L$,

$$[k] = MT^{-2},$$

which is the entry in the table.

These quantities combine into many – infinitely many – dimensionless combinations or groups:

$$\frac{kT^2}{m}, \frac{m}{kT^2}, \left(\frac{kT^2}{m}\right)^{25}, \pi \frac{m}{kT^2}, \dots$$

The groups are redundant. You can construct them from only one group. In fancy terms, all the dimensionless groups are formed from one *independent* dimensionless group. What combination to use for that one group is up to you, but you need only one group. I like kT^2/m .

So any true statement about the period can be written just using kT^2/m . That requirement limits the possible statements to

$$\frac{kT^2}{m} = C,$$

where C is a dimensionless constant. This form has two important consequences:

1. The amplitude x_0 does not affect the period. This independence is also known as simple harmonic motion. The analysis in [Section 5.2](#) gave an approximate argument for why the period should be independent of the amplitude. So that approximate argument turns out to be an exact argument.

2. The constant C is independent of k and m . So I can measure it for one spring–mass system and know it for all spring–mass systems, no matter the mass or spring constant. The constant is a universal constant.

The requirement that dimensions be valid has simplified the analysis of the spring–mass system. Without using dimensions, the problem would be to find (or measure) the three-variable function f that connects m , k , and x_0 to the period:

$$T = f(m, k, x_0).$$

Whereas using dimensions reveals that the problem is simpler: to find the function h such that

$$\frac{kT^2}{m} = h().$$

Here $h()$ means a function of no variables. Why no variables? Because the right side contains all the other quantities on which kT^2/m could depend. However, dimensional analysis says that the variables appear only through the combination kT^2/m , which is already on the left side. So no variables remain to be put on the right side; hence h is a function of zero variables. The only function of zero variables is a constant, so $kT^2/m = C$.

This pattern illustrates a famous quote from the statistician and physicist Harold Jeffreys [25, p. 82]:

A good table of functions of one variable may require a page; that of a function of two variables a volume; that of a function of three variables a bookcase; and that of a function of four variables a library.

Use dimensions; avoid tables as big as a library!

Dimensionless groups are a kind of invariant: They are unchanged even when the system of units is changed. Like any invariant, a dimensionless group is an abstraction (**Chapter 3**). So, looking for dimensionless groups is recipe for developing new abstractions.

6.4 Hydrogen atom

Hydrogen is the simplest atom, and studying hydrogen is the simplest way to understand the **atomic theory**. Feynman has explained the importance of the atomic theory in his famous lectures on physics [, p. 1-2]:

If, in some cataclysm, all of scientific knowledge were to be destroyed, and only one sentence passed on to the next generations

of creatures, what statement would contain the most information in the fewest words? I believe it is the *atomic hypothesis* (or the atomic fact, or whatever you wish to call it) that *all things are made of atoms – little particles that move around in perpetual motion, attracting each other when they are a little distance apart, but repelling upon being squeezed into one another*. In that one sentence, you will see, there is an *enormous* amount of information about the world. . .

The atomic theory was first stated by Democritus. (Early Greek science and philosophy is discussed with wit, sympathy, and insight in Bertrand Russell's *History of Western Philosophy* [30].) Democritus could not say much about the properties of atoms. With modern knowledge of classical and quantum mechanics, and dimensional analysis, you can say more.

6.4.1 Dimensional analysis

The next example of dimensional reasoning is the hydrogen atom in order to answer two questions. The first question is how big is it. That size sets the size of more complex atoms and molecules. The second question is how much energy is needed to disassemble hydrogen. That energy sets the scale for the bond energies of more complex substances, and those energies determine macroscopic quantities like the stiffness of materials, the speed of sound, and the energy content of fat and sugar. All from hydrogen!

The first step in a dimensional analysis is to choose the relevant variables. A simple model of hydrogen is an electron orbiting a proton. The orbital force is provided by electrostatic attraction between the proton and electron. The magnitude of the force is

$$\frac{e^2}{4\pi\epsilon_0} \frac{1}{r^2},$$

where r is the distance between the proton and electron. The list of variables should include enough variables to generate this expression for the force. It could include q , ϵ_0 , and r separately. But that approach is needlessly complex: The charge q is relevant only because it produces a force. So the charge appears only in the combined quantity $e^2/4\pi\epsilon_0$. A similar argument applies to ϵ_0 .

Therefore rather than listing q and ϵ_0 separately, list only $e^2/4\pi\epsilon_0$. And rather than listing r , list a_0 , the common notation for the Bohr radius (the radius of ideal hydrogen). The acceleration of the electron depends on the electrostatic force, which can be constructed from $e^2/4\pi\epsilon_0$ and a_0 , and on its mass m_e . So the list should also include m_e . To find the dimensions of $e^2/4\pi\epsilon_0$, use the formula for force

Var	Dim	What	ω
T^{-1}	frequency	k	
L^{-1}	wavenumber	g	
LT^{-2}	gravity	h	
L	depth	ρ	
ML^{-3}	density	γ	
MT^{-2}	surfacetension		

$$F = \frac{e^2}{4\pi\epsilon_0} \frac{1}{r^2}.$$

Then

$$\left[\frac{e^2}{4\pi\epsilon_0} \right] = [r^2] \times [F] = ML^3T^{-2}.$$

The next step is to make dimensionless groups. However, no combination of these three items is dimensionless. To see why, look at the time dimension because it appears in only one quantity, $e^2/4\pi\epsilon_0$. So that quantity cannot occur in a dimensionless group: If it did, there would be no way to get rid of the time dimensions. From the two remaining quantities, a_0 and m_e , no dimensionless group is possible.

The failure to make a dimensionless group means that hydrogen does not exist in the simple model as we have formulated it. I neglected important physics. There are two possibilities for what physics to add.

One possibility is to add relativity, encapsulated in the speed of light c . So we would add c to the list of variables. That choice produces a dimensionless group, and therefore produces a size. However, the size is not the size of hydrogen. It turns out to be the classical electron radius instead. Fortunately, you do not have to know what the classical electron radius is in order to understand why the resulting size is not the size of hydrogen. Adding relativity to the physics – or adding c to the list – allows radiation. So the orbiting, accelerating electron would radiate. As radiation carries energy away from the electron, it spirals into the proton, meaning that in this world hydrogen does not exist, nor do other atoms.

The other possibility is to add quantum mechanics, which was developed to solve fundamental problems like the existence of matter. The physics of quantum mechanics is complicated, but its effect on dimensional analyses

is simple: It contributes a new constant of nature $\hbar$ whose dimensions are those of angular momentum. Angular momentum is mvr , so

$$[\hbar] = ML^2T^{-1}.$$

The $\hbar$ might save the day. There are now two quantities containing time dimensions. Since $e^2/4\pi\epsilon_0$ has T^{-2} and $\hbar$ has T^{-1} , the ratio $\hbar^2/(e^2/4\pi\epsilon_0)$ contains no time dimensions. Since

$$\left[\frac{\hbar^2}{e^2/4\pi\epsilon_0} \right] = ML,$$

a dimensionless group is

$$\frac{\hbar^2}{\alpha_0 m_e (e^2/4\pi\epsilon_0)}$$

It turns out that all dimensionless groups can be formed from this group. So, as in the spring-mass example, the only possible true statement involving this group is

$$\frac{\hbar^2}{\alpha_0 m_e (e^2/4\pi\epsilon_0)} = \text{dimensionless constant.}$$

Therefore, the size of hydrogen is

$$\alpha_0 \sim \frac{\hbar^2}{m_e (e^2/4\pi\epsilon_0)}.$$

Putting in values for the constants gives

$$\alpha_0 \sim 0.5\text{\AA} = 0.5 \cdot 10^{-10} \text{ m.}$$

It turns out that the missing dimensionless constant is 1, so the dimensional analysis has given the exact answer.

6.4.2 Atomic sizes and substance densities

Hydrogen has a diameter of $1\text{\AA}$. A useful consequence is the rule of thumb is that a typical interatomic spacing is $3\text{\AA}$. This approximation gives a reasonable approximation for the densities of substances, as this section explains.

Var	Dim	What
α_0	L	size
$e^2/4\pi\epsilon_0$	ML^3T^{-2}	
m_e	M	electron mass
$\hbar$	ML^2T^{-1}	quantum

Let A be the atomic mass of the atom; it is (roughly) the number of protons and neutrons in the nucleus. Although A is called a mass, it is dimensionless. Each atom occupies a cube of side length $a \sim 3A$, and has mass Am_{proton} . The density of the substance is

$$\rho = \frac{\text{mass}}{\text{volume}} \sim \frac{Am_{\text{proton}}}{(3A)^3}.$$

You do not need to remember or look up m_{proton} if you multiply this fraction by unity in the form of N_A/N_A , where N_A is Avogadro's number:

$$\rho \sim \frac{Am_{\text{proton}}N_A}{(3A)^3 \times N_A}.$$

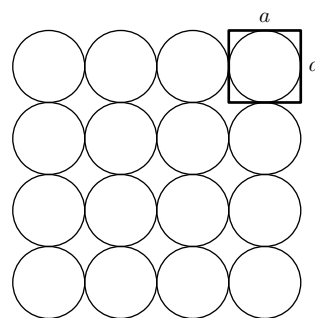
The numerator is A g, because that is how N_A is defined. The denominator is

$$3 \cdot 10^{-23} \text{ cm}^3 \times 6 \cdot 10^{23} = 18.$$

So instead of remembering m_{proton} , you need to remember N_A . However, N_A is more familiar than m_{proton} because N_A arises in chemistry and physics. Using N_A also emphasizes the connection between microscopic and macroscopic values. Carrying out the calculations:

$$\rho \sim \frac{A}{18} \text{ g cm}^{-3}.$$

The table compares the estimate against reality. Most everyday elements have atomic masses between 15 and 150, so the density estimate explains why most densities lie between 1 and 10 g cm^{-3} . It also shows why, for materials physics, cgs units are more convenient than SI units are. A typical cgs density of a solid is 3 g cm^{-3} , and 3 is a modest number and easy to remember and work with. However, a typical SI density of a solid 3000 kg m^{-3} . Numbers such as 3000 are unwieldy. Each time you use it, you have to think, 'How many powers of ten were there again?' So the table tabulates densities using the cgs units of g cm^{-3} . I even threw a joker into the pack – water is not an element! – but the density estimate is amazingly accurate.



<i>Element</i>	$\rho_{\text{estimated}}$	ρ_{actual}
Li	0.39	0.54
H ₂ O	1.0	1.0
Si	1.56	2.4
Fe	3.11	7.9
Hg	11.2	13.5
Au	10.9	19.3
U	13.3	18.7

6.4.3 Physical interpretation

The previous method, dimensional analysis, is mostly mathematical. As a second computation of a_0 , we show you a method that is mostly physics. Besides checking the Bohr radius, it provides a physical interpretation of it. The Bohr radius is the radius of the orbit with the lowest energy (the ground state). The energy is a sum of kinetic and potential energy. This division suggests, again, a divide-and-conquer approach: first the kinetic energy, then the potential energy.

What is the origin of the kinetic energy? The electron does not orbit in any classical sense. If it orbited, it would, as an accelerating charge, radiate energy and spiral into the nucleus. According to quantum mechanics, however, the proton confines the electron to a region of size r – still unknown to us – and the electron exists in a so-called stationary state. The nature of a stationary state is mysterious; no one understands quantum mechanics, so no one understands stationary states except mathematically. However, in an approximate estimate you can ignore details such as the meaning of a stationary state. The necessary information here is that the electron is, as the name of the state suggests, stationary: It does not radiate. The problem then is to find the size of the region to which the electron is confined. In reality the electron is smeared over the whole universe; however, a significant amount of it lives within a typical radius. This typical radius we estimate and call a_0 .

For now let this radius be an unknown r and study how the kinetic energy depends on r . Confinement gives energy to the electron according to the **uncertainty principle**:

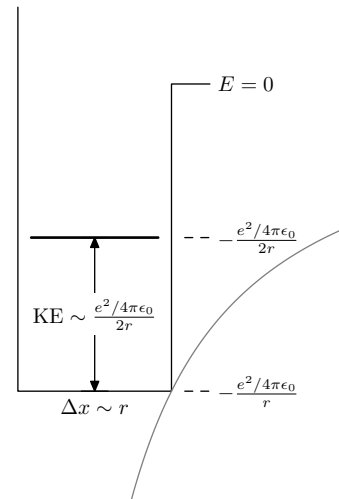
$$\Delta x \Delta p \sim \hbar,$$

where Δx is the position uncertainty and Δp is the momentum uncertainty of the electron. In this model $\Delta x \sim r$, as shown in the figure, so $\Delta p \sim \hbar/r$. The kinetic energy of the electron is

$$E_{\text{Kinetic}} \sim \frac{(\Delta p)^2}{m_e} \sim \frac{\hbar^2}{m_e r^2}.$$

This energy is the **confinement energy** or the **uncertainty energy**. This idea recurs in the book.

The potential energy is the classical expression



$$E_{\text{Potential}} \sim -\frac{e^2}{4\pi\epsilon_0 r}.$$

The total energy is the combination

$$E = E_{\text{Potential}} + E_{\text{Kinetic}} \sim -\frac{e^2}{4\pi\epsilon_0 r} + \frac{\hbar^2}{m_e r^2}.$$

The two energies compete. At small r , kinetic energy wins, because of the $1/r^2$; at large r , potential energy wins, because it goes to zero less rapidly. Is there a minimum combined energy at some intermediate value of r ? There has to be. At small r , the slope dE/dr is negative. At large r , it is positive. At an intermediate r , the slope crosses between positive and negative. The energy is a minimum there. The location would be easy to estimate if the energy were written in dimensionless form. Such a rewriting is not mandatory in this example, but it is helpful in complicated examples and is worth learning in this example.

In constructing the dimensionless group containing a_0 , we constructed another length:

$$l = \frac{\hbar^2}{m_e(e^2/4\pi\epsilon_0)}.$$

To scale any length – to make it dimensionless – divide it by l . So in the total energy the scaled radius

$$\bar{r} \equiv \frac{r}{l}.$$

The other unknown in the total energy is the energy itself. To make it dimensionless, a reasonable energy scale to use is $e^2/4\pi\epsilon_0 l$ by defining scaled energy as

$$\bar{E} \equiv \frac{E}{e^2/4\pi\epsilon_0 l}.$$

Using the dimensionless length and energy, the total energy

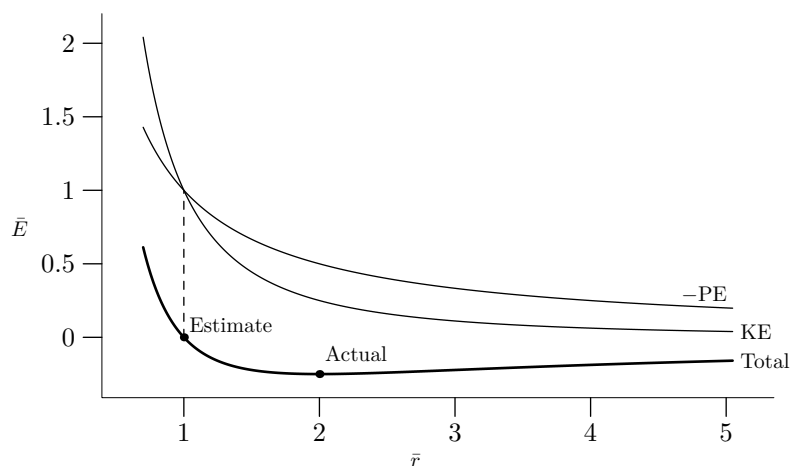
$$E = E_{\text{Potential}} + E_{\text{Kinetic}} \sim -\frac{e^2}{4\pi\epsilon_0 r} + \frac{\hbar^2}{m_e r^2}$$

becomes

$$\bar{E} \sim -\frac{1}{\bar{r}} + \frac{1}{\bar{r}^2}.$$

The ugly constants are placed into the definitions of scaled length and energy. This dimensionless energy is easy to think about and to sketch.

Simple calculus: minimizing scaled energy $\bar{E}$ versus scaled bond length $\bar{r}$. The scaled energy is the sum of potential and kinetic energy. The shape of this energy illustrates Feynman's explanation of the atomic hypothesis. At a 'little distance apart' – for large $\bar{r}$ – the curve slopes upward; to lower their energy, the proton and electron prefer to move closer, and the resulting force is attractive. 'Upon being squeezed into one another' – for small $\bar{r}$ – the potential rapidly increases, so the force between the particles is repulsive. Somewhere between the small and large regions of $\bar{r}$, the force is zero.



Calculus (differentiation) locates this minimum-energy $\bar{r}$ at $\bar{r}_{\min} = 2$. An alternative method is **cheap minimization**: When two terms compete, the minimum occurs when the terms are roughly equal. This method of minimization is familiar from [Section 4.5.2](#).

Equating the two terms $\bar{r}^{-1}$ and $\bar{r}^{-2}$ gives $\bar{r}_{\min} \sim 1$. This result gives a scaled length. In actual units, it is

$$r_{\min} = l\bar{r}_{\min} = \frac{\hbar^2}{m_e(e^2/4\pi\epsilon_0)},$$

which is the Bohr radius computed using dimensional analysis. The sloppiness in estimating the kinetic and potential energies has canceled the error introduced by cheap minimization!

Here is how to justify cheap minimization. Consider a reasonable general form for E :

$$E(r) = \frac{A}{r^n} - \frac{B}{r^m}.$$

This form captures the important feature of the combined energy

$$E = E_{\text{Potential}} + E_{\text{Kinetic}} \sim -\frac{e^2}{4\pi\epsilon_0 r} + \frac{\hbar^2}{m_e r^2},$$

that two terms represent competing physical effects. Mathematically, that physical fact is shown by the opposite signs.

To find the minimum, solve $E'(r_{\min}) = 0$ or

$$-n \frac{A}{r_{\min}^{n+1}} + m \frac{B}{r_{\min}^{m+1}} = 0.$$

The solution is

$$\frac{A}{r_{\min}^n} = \frac{n}{m} \frac{B}{r_{\min}^m} \quad (\text{calculus}).$$

This method minimizes the combined energy by equating the two terms A/r^n and B/r^m :

$$\frac{A}{r_{\min}^n} = \frac{B}{r_{\min}^m}.$$

This approximation lacks the n/m factor in the exact result. The ratio of the two estimates for $r_{\min}$ is

$$\frac{\text{approximate estimate}}{\text{calculus estimate}} \sim \left(\frac{n}{m}\right)^{1/(m-n)},$$

which is smaller than 1 unless $n = m$, when there is no maximum or minimum. So the approximate method underestimates the location of minima and maxima.

To judge the method in practice, apply it to a typical example: the potential between nonpolar atoms or molecules, such as between helium, xenon, or methane. This potential is well approximated by the so-called Lennard-Jones potential where $m = 6$ and $n = 12$:

$$U(r) \sim \frac{A}{r^{12}} - \frac{B}{r^6}.$$

The approximate result underestimates $r_{\min}$ by a factor of

$$\left(\frac{12}{6}\right)^{1/6} \sim 1.15.$$

An error of 15 percent is often small compared to the other inaccuracies in an approximate computation, so this method of approximate minimization is a valuable time-saver.

Now return to the original problem: determining the Bohr radius. The approximate minimization predicts the correct value. Even if the method were not so charmed, there is no point in doing a proper, calculus minimization. The calculus method is *too accurate* given the inaccuracies in the rest of the derivation.

Engineers understand this idea of not over-engineering a system. If a bicycle most often breaks at welds in the frame, there is little point replacing the metal between the welds with expensive, high-strength aerospace materials. The new materials might last 100 years instead of 50 years, but such a replacement would be overengineering. To improve a bicycle, put effort into improving or doing without the welds.

In estimating the Bohr radius, the kinetic-energy estimate uses a crude form of the uncertainty principle, $\Delta p \Delta x \sim \hbar$, whereas the true statement is that $\Delta p \Delta x \geq \hbar/2$. The estimate also uses the approximation $E_{\text{kinetic}} \sim (\Delta p)^2/m$. This approximation contains m instead of $2m$ in the denominator. It also assumes that Δp can be converted into an energy as though it were a true momentum rather than merely a crude estimate for the root-mean-square momentum. The potential- and kinetic-energy estimates use a crude definition of position uncertainty Δx : that $\Delta x \sim r$. After making so many approximations, it is pointless to minimize the result using the elephant gun of differential calculus. The approximate method is as accurate as, or perhaps more accurate than the approximations in the energy.

This method of equating competing terms is **balancing**. We balanced the kinetic energy against the potential energy by assuming that they are roughly the same size. The consequence is that

$$a_0 \sim \frac{\hbar^2}{m_e(e^2/4\pi\epsilon_0)}.$$

Nature could have been unkind: The potential and kinetic energies could have differed by a factor of 10 or 100. But Nature is kind: The two energies are roughly equal, except for a constant that is nearly 1. ‘Nearly 1’ is also called **of order unity**. This rough equality occurs in many examples, and you often get a reasonable answer by pretending that two energies (or two quantities with the same units) are equal. When the quantities are potential and kinetic energy, as they often are, you get extra safety: The so-called virial theorem protects you against large errors (for more on the virial theorem, see any intermediate textbook on classical dynamics).

6.5 Bending of light by gravity

Rocks, birds, and people feel the effect of gravity. So why not light? The analysis of that question is a triumph of Einstein's theory of general relativity. I can calculate how gravity bends light by solving the so-called geodesic equations from general relativity:

$$\frac{d^2 x^\beta}{d\lambda^2} + \Gamma_{\mu\nu}^\beta \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0.$$

To compute the Christoffel symbols $\Gamma_{\mu\nu}^\beta$ requires solving for the metric tensor $g_{\mu\nu}$, which requires solving the curvature equations $R_{\mu\nu} = 0$.

The curvature equations are a shorthand for ten partial-differential equations. The equations are rich in mathematical interest but are a nightmare to solve. The equations are numerous – that's one problem – but worse, they are not linear. So the standard trick, which is to guess a type of solution and form new solutions by combining the basic types, does not work. You can spend a decade learning advanced mathematics to solve the equations exactly. Or you can accept the great principle of analysis: When the going gets tough, lower your standards. If I sacrifice accuracy, I can explain light bending in less than one thousand pages using mathematics and physics that you (and I!) already know.

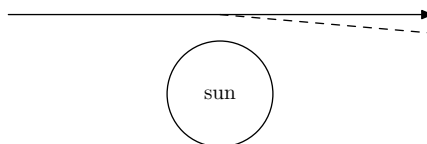
The simpler method is dimensional analysis, in the usual three steps:

1. Find the relevant parameters.
2. Find dimensionless groups.
3. Use the groups to make the most general dimensionless statement.
4. Add physical knowledge to narrow the possibilities.

The following sections do each step.

6.5.1 Finding parameters

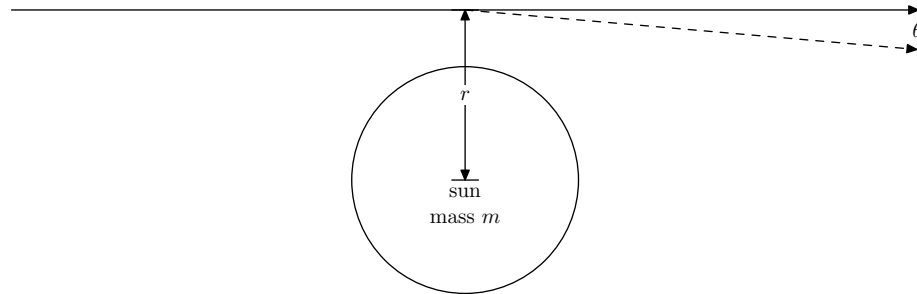
The first step in a dimensional analysis is to decide what physical parameters the bending angle can depend on. An unlabeled diagram prods me into thinking of labels, many of which are parameters of the problem:



Here are reasons to include various parameters:

1. The list has to include the quantity to solve for. So the angle θ is the first item in the list.
2. The mass of the sun, m , has to affect the angle. Black holes greatly deflect light, probably because of their huge mass.
3. A faraway sun or black hole cannot strongly affect the path (near the earth light seems to travel straight, in spite of black holes all over the universe); therefore r , the distance from the center of the mass, is a relevant parameter. The phrase 'distance from the center' is ambiguous, since the light is at various distances from the center. Let r be the distance of closest approach.
4. The dimensional analysis needs to know that gravity produces the bending. The parameters listed so far do not create any forces. So include Newton's gravitational constant G .

Here is the same diagram with important parameters labeled:



Here is a table of the parameters and their dimensions:

Parameter	Meaning	Dimensions
θ	angle	—
m	mass of sun	M
G	Newton's constant	$L^3 T^{-2} M^{-1}$
r	distance from center of sun	L

where, as you might suspect, L , M , and T represent the dimensions of length, mass, and time, respectively.

6.5.2 Dimensionless groups

What are the dimensionless groups? The parameter θ is an angle, which is already dimensionless. The other variables, G , m , and r , cannot form a second dimensionless group. To see why, following the dimensions of mass. It appears only in G and m , so a dimensionless group would contain the

product Gm , which has no mass dimensions in it. But Gm and r cannot get rid of the time dimensions. So there is only one independent dimensionless group, for which θ is the simplest choice.

I want a second dimensionless group because otherwise my analysis seems like nonsense. Any physical solution can be written in dimensionless form; this idea is the foundation of dimensional analysis. With only one dimensionless group, θ , I have to conclude that θ depends on no variables at all:

$$\theta = \text{function of other dimensionless groups,}$$

but there are no other dimensionless groups, so

$$\theta = \text{constant.}$$

This conclusion is crazy! The angle must depend on at least one of m and r . My physical picture might be confused, but it's not so confused that neither variable is relevant. So I need to make another dimensionless group on which θ can depend. Therefore, I return to Step 1: Finding parameters.

The list so far lacks a crucial parameter.

What physics have I neglected? Free associating often suggests the missing parameter. Unlike rocks, light is difficult to deflect, otherwise humanity would not have waited until the 1800s to study the deflection, whereas the path of rocks was studied at least as far back as Aristotle and probably for millions of years beforehand. Light travels much faster than rocks, which may explain why light is so difficult to deflect: The gravitational field 'gets hold of it' only for a short time. But none of my parameters distinguish between light and rocks. Therefore I should include c , the speed of light. It introduces the fact that I'm studying light, and it does so with a useful distinguishing parameter, the speed.

Here is the latest table of parameters and dimensions:

<i>Parameter</i>	<i>Meaning</i>	<i>Dimensions</i>
θ	angle	—
m	mass of sun	M
G	Newton's constant	$L^3T^{-2}M^{-1}$
r	distance from center of sun	L
c	speed of light	LT^{-1}

Length is strewn all over the parameters (it's in G , r , and c). Mass, however, appears in only G and m , so I already know I need a combination such as Gm to cancel out mass. Time also appears in only two parameters: G and

c. To cancel out time, I need to form Gm/c^2 . This combination has one length in it, so a dimensionless group is Gm/rc^2 .

6.5.3 Drawing conclusions

The most general relation between the two dimensionless groups is

$$\theta = f\left(\frac{Gm}{rc^2}\right).$$

Dimensional analysis cannot tell me the correct function f .

Physical reasoning and symmetry narrow the possibilities. First, strong gravity – from a large G or m – should increase the angle. So f should be an increasing function. Now try symmetry: Imagine a world where gravity is repulsive or, equivalently, the gravitational constant is negative. Then the angle should also be negative, so f should be an odd function. This symmetry argument eliminates choices like $f(Gm/rc^2) \sim (Gm/rc^2)^2$.

The simplest guess is that f is the identity function. Then the bending angle is

$$\theta = \frac{Gm}{rc^2}.$$

There is likely a dimensionless constant in f :

$$\theta = 7 \frac{Gm}{rc^2}$$

or

$$\theta = 0.3 \frac{Gm}{rc^2}$$

are also possible. This freedom means

$$\theta \sim \frac{Gm}{rc^2}.$$

6.5.4 Comparison with exact calculations

Different theories of gravity give the same result

$$\theta \sim \frac{Gm}{rc^2};$$

the only variation is in the value for the missing dimensionless constant. Here are those values from exact calculation:

$$\theta = \frac{Gm}{rc^2} \times \begin{cases} 1 & \text{(simplest guess);} \\ 2 & \text{(Newtonian gravity);} \\ 4 & \text{(Einstein's theory).} \end{cases}$$

Here is a rough explanation of the origin of those constants. The 1 for the simplest guess is just that. The 2 for Newtonian gravity is from integrating angular factors like cosine and sine that determine the position of the photon as it moves toward and past the sun.

The most interesting constant is the 4 for general relativity, which is twice the Newtonian value because light moves at the speed of light. The extra bending is a consequence of Einstein's theory of special relativity putting space and time on the same level. The theory of general relativity then formulates gravity in terms of the curvature of spacetime. Newton's theory is the limit of general relativity that considers only time curvature; general relativity itself also calculates the space curvature. Since most objects move much slower than the speed of light, meaning that they travel much farther in time than in space, they feel mostly the time curvature. The Newtonian analysis is fine for those objects. Since light moves at the speed of light, it sees equal amounts of space and time curvature, so it bends twice as far as the Newtonian theory would predict.

6.5.5 Numbers!

At the surface of the Earth, the strength is

$$\frac{Gm}{rc^2} \sim \frac{6.7 \cdot 10^{-11} \text{ m}^3 \text{ s}^{-2} \text{ kg}^{-1} \times 6.0 \cdot 10^{24} \text{ kg}}{6.4 \cdot 10^6 \text{ m} \times 3.0 \cdot 10^8 \text{ m s}^{-1} \times 3.0 \cdot 10^8 \text{ m s}^{-1}} \sim 10^{-9}.$$

This miniscule value is the bending angle (in radians). So if physicists want to show that light bends, they had better look beyond the earth! That statement is based on another piece of dimensional analysis and physical reasoning, whose result I quote without proof: A telescope with mirror of diameter d can resolve angles roughly as small as λ/d , where λ is the wavelength of light. One way to measure the bending of light is to measure the change in position of the stars. A lens that could resolve an angle of 10^{-9} has a diameter of at least

$$d \sim \lambda/\theta \sim \frac{0.5 \cdot 10^{-6} \text{ m}}{10^{-9}} \sim 500 \text{ m}.$$

Large lenses warp and crack; one of the largest lenses made is 6 m. So there is no chance of detecting an angle of 10^{-9} .

Physicists therefore searched for another source of light bending. In the solar system, the largest mass is the sun. At the surface of the sun, the field strength is

$$\frac{Gm}{rc^2} \sim \frac{6.7 \cdot 10^{-11} \text{ m}^3 \text{ s}^{-2} \text{ kg}^{-1} \times 2.0 \cdot 10^{30} \text{ kg}}{7.0 \cdot 10^8 \text{ m} \times 3.0 \cdot 10^8 \text{ m s}^{-1} \times 3.0 \cdot 10^8 \text{ m s}^{-1}} \sim 2.1 \cdot 10^{-6} \approx 0.4''.$$

This angle, though small, is possible to detect: The required lens diameter is roughly

$$d \sim \lambda/\theta \sim \frac{0.5 \cdot 10^{-6} \text{ m}}{2.1 \cdot 10^{-6}} \sim 20 \text{ cm}.$$

The eclipse expedition of 1919, led by Arthur Eddington of Cambridge, tried to measure exactly this effect.

For many years Einstein believed that his theory of gravity would predict the Newtonian value, which turns out to be 0.87 arcseconds for light just grazing the surface of the sun. The German mathematician, Soldner, derived the same result in 1803. Fortunately for Einstein's reputation, the eclipse expeditions that went to test his (and Soldner's) prediction got rained or clouded out. By the time an expedition got lucky with the weather (Eddington's in 1919), Einstein had invented a new theory of gravity, which predicted 1.75 arcseconds. The goal of Eddington's expedition was to decide between the Newtonian and general relativity values. The measurements are difficult, and the results were not accurate enough to decide which theory was right. But 1919 was the first year after the World War, in which Germany and Britain had fought each other almost to oblivion. A theory invented by a German, confirmed by an Englishman (from Newton's university, no less) – such a picture was comforting after the trauma of war, so the world press and scientific community saw what they wanted to: Einstein vindicated! A proper confirmation of Einstein's prediction came only with the advent of radio astronomy, which could measure small deflections accurately. I leave you with this puzzle: If the accuracy of a telescope is λ/d , how could radio telescopes be more accurate than optical ones, since radio waves have a longer wavelength than light has?!

6.6 Buckingham Pi theorem

The second step in a dimensional analysis is to make dimensionless groups. That task is simpler by knowing in advance how many groups to look for. The Buckingham Pi theorem provides that number. I derive it with a series of examples.

Here is a possible beginning of the theorem statement: *The number of dimensionless groups is...* Try it on the light-bending example. How many groups can the variables θ , G , m , r , and c produce? The possibilities include θ , θ^2 , Gm/rc^2 , $\theta Gm/rc^2$, and so on. The possibilities are infinite! Now apply the theorem statement to estimating the size of hydrogen, before including quantum mechanics in the list of variables. That list is a_0 (the size), $e^2/4\pi\epsilon_0$, and m_e . That list produces no dimensionless groups. So it seems that the number of groups would be zero – if no groups are possible – or infinity, if even one group is possible.

Here is an improved theorem statement taking account of the redundancy: *The number of independent dimensionless groups is...* To complete the statement, try a few examples:

1. Bending of light. The five quantities θ , G , m , r , and c produce two independent groups. A convenient choice for the two groups is θ and Gm/rc^2 , but any other independent set is equally valid, even if not as intuitive.
2. Size of hydrogen without quantum mechanics. The three quantities a_0 (the size), $e^2/4\pi\epsilon_0$, and m_e produce zero groups.
3. Size of hydrogen with quantum mechanics. The four quantities a_0 (the size), $e^2/4\pi\epsilon_0$, m_e , and $\hbar$ produce one independent group.

These examples fit a simple pattern:

$$\text{no. of independent groups} = \text{no. of quantities} - 3.$$

The 3 is a bit distressing because it is a magic number with no explanation. It is also the number of basic dimensions: length, mass, and time. So perhaps the statement is

$$\text{no. of independent groups} = \text{no. of quantities} - \text{no. of dimensions}.$$

Test this statement with additional examples:

1. Period of a spring-mass system. The quantities are T (the period), k , m , and x_0 (the amplitude). These four quantities form one independent dimensionless group, which could be kT^2/m . This result is consistent with the proposed theorem.
2. Period of a spring-mass system (without x_0). Since the amplitude x_0 does not affect the period, the quantities could have been T (the period), k , and m . These three quantities form one independent dimensionless group, which again could be kT^2/m . This result is also consistent with

the proposed theorem, since T , k , and m contain only two dimensions (mass and time).

The theorem is safe until we try to derive Newton's second law. The force F depends on mass m and acceleration a . Those three quantities contain three dimensions – mass, length, and time. Three minus three is zero, so the proposed theorem predicts zero independent dimensionless groups. Whereas $F = ma$ tells me that F/ma is a dimensionless group.

This problem can be fixed by adding one word. Look at the dimensions of F , m , and a . All the dimensions – M or MLT^{-2} or LT^{-2} – can be constructed from only *two* dimensions: M and LT^{-2} . The key idea is that the original set of three dimensions are not independent, whereas the pair M and LT^{-2} are independent. So:

Var	Dim	What
F	MLT^{-2}	force
m	M	mass
a	LT^{-2}	acceleration

no. of independent groups = no. of quantities – no. of *independent* dimensions.

And that statement is the Buckingham Pi theorem [3].

More on dimensionless groups

Summary:

Further reading:

Problem 6.1 Counting dimensionless groups

How many independent dimensionless groups are there in the following sets of variables:

- a. size of hydrogen including relativistic effects:

$$e^2/4\pi\epsilon_0, \hbar, c, a_0 \text{ (Bohr radius)}, m_e \text{ (electron mass)}.$$

- b. period of a spring–mass system in a gravitational field:

$$T \text{ (period)}, k \text{ (spring constant)}, m, x_0 \text{ (amplitude)}, g.$$

- c. speed at which a free-falling object hits the ground:

$$v, g, h \text{ (initial drop height)}.$$

- d. [tricky!] weight W of an object:

$W, g, m.$

Problem 6.2 Integrals by dimensions

You can use dimensions to do integrals. As an example, try this integral:

$$I(\beta) = \int_{-\infty}^{\infty} e^{-\beta x^2} dx.$$

Which choice has correct dimensions: (a.) $\sqrt{\pi}\beta^{-1}$ (b.) $\sqrt{\pi}\beta^{-1/2}$ (c.) $\sqrt{\pi}\beta^{1/2}$ (d.) $\sqrt{\pi}\beta^1$

Hints:

1. The dimensions of dx are the same as the dimensions of x .
2. Pick interesting dimensions for x , such as length. (If x is dimensionless then you cannot use dimensional analysis on the integral.)

Problem 6.3 How to avoid remembering lots of constants

Many atomic problems, such as the size or binding energy of hydrogen, end up in expressions with $\hbar$, the electron mass m_e , and $e^2/4\pi\epsilon_0$, which is a nicer way to express the squared electron charge. You can avoid having to remember those constants if instead you remember these values instead:

$$\begin{aligned}\hbar c &\approx 200 \text{ eV nm} = 2000 \text{ eV \AA} \\ m_e c^2 &\sim 0.5 \cdot 10^6 \text{ eV} \\ \frac{e^2/4\pi\epsilon_0}{\hbar c} &\equiv \alpha \approx \frac{1}{137} \quad (\text{fine-structure constant}).\end{aligned}$$

Use those values to evaluate the Bohr radius in angstroms ($1 \text{ \AA} = 0.1 \text{ nm}$):

$$a_0 = \frac{\hbar^2}{m_e(e^2/4\pi\epsilon_0)}.$$

As an example calculation using the $\hbar c$ value, here is the energy of a photon:

$$E = hf = 2\pi\hbar f = 2\pi\hbar \frac{c}{\lambda},$$

where f is its frequency and λ is its wavelength. For green light, $\lambda \sim 600 \text{ nm}$, so

$$E \sim \frac{\overbrace{2\pi}^{\text{2}\pi} \times \overbrace{200 \text{ eV nm}}^{\hbar c}}{\underbrace{600 \text{ nm}}_{\lambda}} \sim 2 \text{ eV}.$$

Problem 6.4 Heavy nuclei

In lecture we analyzed hydrogen, which is one electron bound to one proton. In this problem you study the innermost electron in an atom such as uranium that has many protons, and analyze one physical consequence of its binding energy.

So, imagine a nucleus with Z protons around which orbits one electron. Let $E(Z)$ be the binding energy (the hydrogen energy is the case $Z = 1$).

- Show that the ratio $E(Z)/E(1)$ is Z^2 .
- In lecture, we derived that $E(1)$ is the kinetic energy of an electron moving with speed αc where α is the fine-structure constant (roughly 10^{-2}). How fast does the innermost electron move around a heavy nucleus with charge Z ?
- When that speed is comparable to the speed of light, the electron has a kinetic energy comparable to its (relativistic) rest energy. One consequence of such a high kinetic energy is that the electron has enough kinetic energy to produce a positron (an anti-electron) out of nowhere ('pair creation'). That positron leaves the nucleus, turning a proton into a neutron as it exits. So the atomic number Z drops by one: The nucleus is unstable! Relativity sets an upper limit for Z .

Estimate that maximum Z and compare it with the Z for the heaviest stable nucleus (uranium).

Problem 6.5 Power radiated by an accelerating charge

Electromagnetism, where the usual derivations are so cumbersome, is an excellent area to apply dimensional analysis. In this problem you work out the power radiated by an accelerating charge, which is how radio stations work.

So, consider a particle with charge q , with position x , velocity v , and acceleration a .

- What variables are relevant to the radiated power P ? The position cannot matter because it depends on the origin of the coordinate system, whereas the power radiated cannot depend on the origin. The velocity cannot matter because of relativity: You can transform to a reference frame where $v = 0$, but that change will not affect the radiation (otherwise you could distinguish a moving frame from a non-moving frame, in violation of the principle of relativity). So the acceleration a is all that's left to determine the radiated power. [This line of argument is slightly dodgy, but it works for low speeds.]

- Using P , $q^2/4\pi\epsilon_0$, and a , how many dimensionless quantities can you form?
- Fix the problem in the previous part by adding one quantity to the list of variables, and give a physical reason for including the quantity.
- With the new list, use dimensionless groups to find the power radiated by an accelerating point charge. In case you are curious, the exact result contains a dimensionless factor of $2/3$; dimensional analysis triumphs again!

Problem 6.6 Yield from an atomic bomb

Geoffrey Taylor, a famous Cambridge fluid mechanic, annoyed the US government by doing the following analysis. The question he answered: 'What was the yield (in kilotons of TNT) of the first atomic blast (in the New Mexico desert in 1945)?' Declassified pictures, which even had a scale bar, gave the following data on the radius of the explosion at various times:

t (ms)	R (m)
3.26	59.0
4.61	67.3
15.0	106.5
62.0	185.0

- Use dimensional analysis to work out the relation between radius R , time t , blast energy E , and air density ρ .
- Use the data in the table to estimate the blast energy E (in Joules).
- Convert that energy to kilotons of TNT. One gram of TNT releases 1 kcal or roughly 4 kJ.

The actual value was 20 kilotons, a classified number when Taylor published his result ['The Formation of a Blast Wave by a Very Intense Explosion. II. The Atomic Explosion of 1945.', *Proceedings of the Royal Society of London. Series A, Mathematical and Physical* **201**(1065): 175–186 (22 March 1950)]

Problem 6.7 Atomic blast: A physical interpretation

Use energy densities and sound speeds to make a rough physical explanation of the result in the 'yield from an atomic bomb' problem.

Problem 6.8 Rolling down the plane

Four objects, made of identical steel, roll down an inclined plane without slipping. The objects are:

- a large spherical shell,
- a large disc,
- a small solid sphere,
- a small ring.

The large objects have three times the radius of the small objects. Rank the objects by their acceleration (highest acceleration first).

Check your results with exact calculation or with a home experiment.

Problem 6.9 Blackbody radiation

A hot object – a so-called blackbody – radiates energy, and the flux F depends on the temperature T . In this problem you derive the connection using dimensional analysis. The goal is to find F as a function of T . But you need more quantities.

- a. What are the dimensions of flux?
 - b. What two constants of nature should be included because blackbody radiation depends on the quantum theory of radiation?
 - c. What constant of nature should be included because you are dealing with temperature?
 - d. After doing the preceding parts, you have five variables. Explain why these five variables produce one dimensionless group, and use that fact to deduce the relation between flux and temperature.
 - e. Look up the Stefan–Boltzman law and compare your result to it.
-

Part 3

Lossy compression

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Chapter 7

Special cases

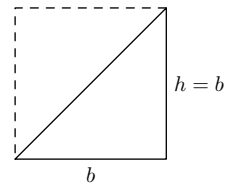
7.1 Pyramid volume

I have been promising to explain the factor of one-third in the volume of a pyramid:

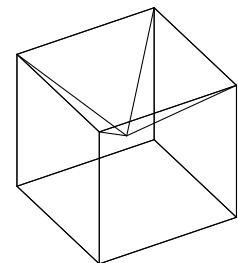
$$V = \frac{1}{3}hb^2.$$

Although the method of special cases mostly cannot explain a dimensionless constant, the volume of a pyramid provides a rare counterexample.

I first explain the key idea in fewer dimensions. So, instead of immediately explaining the one-third in the volume of a pyramid, which is a difficult three-dimensional problem, first find the corresponding constant in a two-dimensional problem. That problem is the area of a triangle with base b and height h : The area is $A \sim bh$. What is the constant? Choose a convenient triangle, perhaps a 45-degree right triangle where $h = b$. Two of those triangles form a square with area b^2 , so $A = b^2/2$ when $h = b$. The constant in $A \sim bh$ is therefore $1/2$ *no matter what b and h are*, so $A = bh/2$.



Now use the same construction in three dimensions. What square-based pyramid, when combined with itself perhaps several times, makes a familiar shape? Only the aspect ratio h/b matters in the following discussion. So choose b conveniently, and then choose h to make a pyramid with the clever aspect ratio. The goal shape is suggested by the square pyramid base. Another solid with the same base is a cube.



Perhaps several pyramids can combine into a cube of side b . To simplify the upcoming arithmetic, I choose $b = 2$. What should the height h be? To decide, imagine how the cube will be constructed. Each

cube has six faces, so six pyramids might make a cube where each pyramid base forms one face of the cube, and each pyramid tip faces inward, meeting in the center of the cube. For the tips to meet in the center of the cube, the height must be $h = 1$. So six pyramids with $b = 2$, and $h = 1$ make a cube with side length 2.

The volume of one pyramid is one-sixth of the volume of the cube:

$$V = \frac{\text{cube volume}}{6} = \frac{8}{6} = \frac{4}{3}.$$

The volume of the pyramid is $V \sim hb^2$, and the missing constant must make volume $4/3$. Since $hb^2 = 4$ for these pyramids, the missing constant is $1/3$. Voilà:

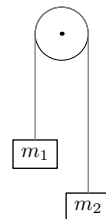
$$V = \frac{1}{3}hb^2 = \frac{4}{3}.$$

7.2 Mechanics

7.2.1 Atwood machine

The next problem illustrates dimensional analysis and special cases in a physical problem. Many of the ideas and methods from the geometry example transfer to this problem, and it introduces more methods and ways of reasoning.

The problem is a staple of first-year physics: Two masses, m_1 and m_2 , are connected and, thanks to a pulley, are free to move up and down. What is the acceleration of the masses and the tension in the string? You can solve this problem with standard methods from first-year physics, which means that you can check the solution that we derive using dimensional analysis, educated guessing, and a feel for functions.



The first problem is to find the acceleration of, say, m_1 . Since m_1 and m_2 are connected by a rope, the acceleration of m_2 is, depending on your sign convention, either equal to m_1 or equal to $-m_1$. Let's call the acceleration a and use dimensional analysis to guess its form. The first step is to decide what variables are relevant. The acceleration depends on gravity, so g should be on the list. The masses affect the acceleration, so m_1 and m_2 are on the list. And that's it. You might wonder what happened to the tension: Doesn't it affect the acceleration? It does, but it is itself a consequence of m_1 , m_2 , and g . So adding tension to the list does not add information; it would instead make the dimensional analysis difficult.

These variables fall into two pairs where the variables in each pair have the same dimensions. So there are two dimensionless groups here ripe for picking: $G_1 = m_1/m_2$ and $G_2 = a/g$. You can make any dimensionless group using these two obvious groups, as experimentation will convince you. Then, following the usual pattern,

Var	Dim	What
a	LT^{-2}	accel. of m_1
g	LT^{-2}	gravity
m_1	M	block mass
m_2	M	block mass

$$\frac{a}{g} = f\left(\frac{m_1}{m_2}\right),$$

where f is a dimensionless function.

Pause a moment. The more thinking that you do to choose a clean representation, the less algebra you do later. So rather than find f using m_1/m_2 as the dimensionless group, first choose a better group. The ratio m_1/m_2 does not respect the symmetry of the problem in that only the sign of the acceleration changes when you interchange the labels m_1 and m_2 . Whereas m_1/m_2 turns into its reciprocal. So the function f will have to do lots of work to turn the unsymmetric ratio m_1/m_2 into a symmetric acceleration.

Back to the drawing board for how to fix G_1 . Another option is to use $m_1 - m_2$. Wait, the difference is not dimensionless! I fix that problem in a moment. For now observe the virtue of $m_1 - m_2$. It shows a physically reasonable symmetry under mass interchange: $G_1 \rightarrow -G_1$. To make it dimensionless, divide it by another mass. One candidate is m_1 :

$$G_1 = \frac{m_1 + m_2}{m_1}.$$

That choice, like dividing by m_2 , abandons the beloved symmetry. But dividing by $m_1 + m_2$ solves all the problems:

$$G_1 = \frac{m_1 - m_2}{m_1 + m_2}.$$

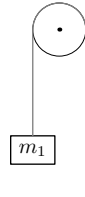
This group is dimensionless and it respects the symmetry of the problem.

Using this G_1 , the solution becomes

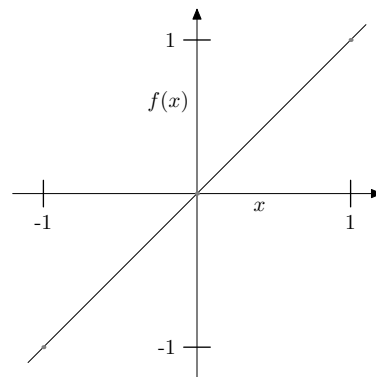
$$\frac{a}{g} = f\left(\frac{m_1 - m_2}{m_1 + m_2}\right),$$

where f is another dimensionless function.

To guess $f(x)$, where $x = G_1$, try special cases. First imagine that m_1 becomes huge. A quantity with mass cannot be huge on its own, however. Here huge means *huge relative to* m_2 , whereupon $x \approx 1$. In this thought experiment, m_1 falls as if there were no m_2 so $a = -g$. Here we've chosen a sign convention with positive acceleration being upward. If m_2 is huge relative to m_1 , which means $x = -1$, then m_2 falls like a stone pulling m_1 upward with acceleration $a = g$. A third limiting case is $m_1 = m_2$ or $x = 0$, whereupon the masses are in equilibrium so $a = 0$.



Here is a plot of our knowledge of f :



The simplest conjecture – an educated guess – is that $f(x) = x$. Then we have our result:

$$\frac{a}{g} = \frac{m_1 - m_2}{m_1 + m_2}.$$

Look how simple the result is when derived in a symmetric, dimensionless form using special cases!

7.3 Drag

Pendulum motion is not a horrible enough problem to show the full benefit of dimensional analysis. Instead try fluid mechanics – a subject notorious for its mathematical and physical complexity; Chandrasekhar's books [6, 7] or the classic textbook of Lamb [19] show that the mathematics is not for the faint of heart.

The next examples illustrate two extremes of fluid flow: oozing and turbulent. An example of oozing flow is ions transporting charge in seawater (Section 7.3.6). An example of turbulent flow is a raindrop falling from the sky after condensing out of a cloud (Section 7.3.7).

To find the terminal velocity, solve the partial-differential equations of fluid mechanics for the incompressible flow of a Newtonian fluid:

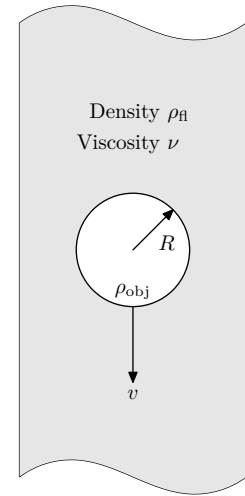
$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{v}, \quad (3 \text{ eqns})$$

$$\nabla \cdot \mathbf{v} = 0. \quad (1 \text{ eqn})$$

Here $\mathbf{v}$ is the fluid velocity, ρ is the fluid density, ν is the kinematic viscosity, and p is the pressure. The first equation is a vector shorthand for three equations, so the full system is four equations.

All the equations are partial-differential equations and three are nonlinear. Worse, they are coupled: Quantities appear in more than one equation. So we have to solve a system of coupled, nonlinear, partial-differential equations. This solution must satisfy boundary conditions imposed by the marble or raindrop. As the object moves, the boundary conditions change. So until you know how the object moves, you do not know the boundary conditions. Until you know the boundary conditions, you cannot find the motion of the fluid or of the object. This coupling between the boundary conditions and solution compounds the difficulty of the problem. It requires that you solve the equations and the boundary conditions together. If you ever get there, then you take the limit $t \rightarrow \infty$ to find the terminal velocity.

Sleep easy! I wrote out the Navier–Stokes equations only to scare you into using dimensional analysis and special-cases reasoning. The approximate approach is easier than solving nonlinear partial-differential equations.



7.3.1 Naive dimensional analysis

To use dimensional analysis, follow the usual steps: Choose relevant variables, form dimensionless groups from them, and solve for the terminal velocity. In choosing quantities, do not forget to include the variable for which you are solving, which here is v . To decide on the other quantities, split them into three categories (divide and conquer):

1. characteristics of the fluid,
2. characteristics of the object, and
3. characteristics of whatever makes the object fall.

The last category is the easiest to think about, so deal with it first. Gravity makes the object fall, so g is on the list.

Consider next the characteristics of the object. Its velocity, as the quantity for which we are solving, is already on the list. Its mass m affects the terminal velocity: A feather falls more slowly than a rock does. Its radius r probably affects the terminal velocity. Instead of listing r and m together, remix them and use r and ρ_{obj} . The two alternatives r and m or r and ρ_{obj} provide the same information as long as the object is uniform: You can compute ρ_{obj} from m and r and can compute m from ρ_{obj} and r .

Choose the preferable pair by looking ahead in the derivation. The relevant properties of the fluid include its density ρ_{fl} . If the list also includes ρ_{obj} , then the results might contain pleasing dimensionless ratios such as ρ_{obj}/ρ_{fl} (a dimensionless group!). The ratio ρ_{obj}/ρ_{fl} has a more obvious physical interpretation than a combination such as $m/\rho_{fl}r^3$, which, except for a dimensionless constant, is more obscurely the ratio of object and fluid densities. So choose ρ_{obj} and r over m and r .

Scaling arguments also favor the pair ρ_{obj} and r . In a scaling argument you imagine varying, say, a size. Size, like heat, is an extensive quantity: a quantity related to amount of stuff. When you vary the size, you want as few other variables as possible to change so that those changes do not obscure the effect of changing size. Therefore, whenever possible replace extensive quantities with **intensive quantities** like temperature or density. The pair m and r contains two extensive quantities, whereas the preferable pair ρ_{obj} and r contains only one extensive quantity.

Now consider properties of the fluid. Its density ρ_{fl} affects the terminal velocity. Perhaps its viscosity is also relevant. Viscosity measures the tendency of a fluid to reduce velocity differences in the flow. You can observe an analog of viscosity in traffic flow on a multilane highway. If one lane moves much faster than another, drivers switch from the slower to the faster lane, eventually slowing down the faster lane. Local decisions of the drivers reduce the velocity gradient. Similarly, molecular motion (in a gas) or collisions (in a fluid) transports speed (really, momentum) from fast- to slow-flowing regions. This transport reduces the velocity difference between the regions. Oozier (more viscous) fluids probably produce

more drag than thin fluids do. So viscosity belongs on the list of relevant variables.

Fluid mechanicians have defined two viscosities: dynamic viscosity η and kinematic viscosity ν . [Sadly, we could not use the mellifluous term *fluid mechanics* to signify a host of physicists agonizing over the equations of fluid mechanics; it would not distinguish the toilers from their toil.] The two viscosities are related by $\eta = \rho_{fl}\nu$. *Life in Moving Fluids* [36, pp. 23–25] discusses the two types of viscosity in detail. For the analysis of drag force, you need to know only that viscous forces are proportional to viscosity. Which viscosity should we use? Dynamic viscosity hides ρ_{fl} inside the product $\nu\rho_{fl}$; a ratio of ρ_{obj} and η then looks less dimensionless than it is because ρ_{obj} 's partner ρ_{fl} is buried inside η . Therefore the kinematic viscosity ν usually gives the more insightful results. Summarizing the discussion, the table lists the variables by category.

The next step is to find dimensionless groups. The Buckingham Pi theorem (Section 6.6) says that the six variables and three independent dimensions result in three dimensionless groups.

Before finding the groups, consider the consequences of three groups. Three?! Three dimensionless groups produce this form for the terminal velocity v :

Var	Dim	What
ν	L^2T^{-1}	kinematic viscosity
ρ_{fl}	ML^{-3}	fluid density
r	L	object radius
v	LT^{-1}	terminal velocity
ρ_{obj}	ML^{-3}	object density
g	LT^{-2}	gravity

group with $v = f(\text{other group 1, other group 2})$.

To deduce the properties of f requires physics knowledge. However, studying a two-variable function is onerous. A function of one variable is represented by a curve and can be graphed on a sheet of paper. A function of two variables is represented by a surface. For a complete picture it needs three-dimensional paper (do you have any?); or you can graph many slices of it on regular two-dimensional paper. Neither choice is appealing. This brute-force approach to the terminal velocity produces too many dimensionless groups.

If you simplify only after you reach the complicated form

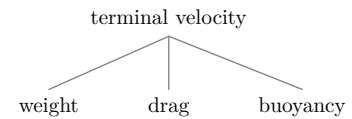
group with $v = f(\text{other group 1, other group 2})$,

you carry baggage that you eventually discard. When going on holiday to the Caribbean, why pack skis that you never use but just cart around everywhere? Instead, at the beginning of the analysis, incorporate the physics

knowledge. That way you simplify the remainder of the derivation. To follow this strategy of packing light – of packing only what you need – consider the physics of terminal velocity in order to make simplifications now.

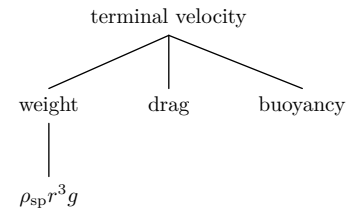
7.3.2 Simpler approach

The adjective *terminal* in the phrase ‘terminal velocity’ hints at the physics that determines the velocity. Here ‘terminal’ is used in its sense of final, as in after an infinite time. It indicates that the velocity has become constant, which happens only when no net force acts on the marble. This line of thought suggests that we imagine the forces acting on the object: gravity, buoyancy, and drag. The terminal velocity is velocity at which the drag, gravitational, and buoyant forces combine to make zero net force. Divide-and-conquer reasoning splits the terminal-velocity problem into three simpler problems.



The gravitational force, also known as the weight, is mg . Instead of m we use $(4\pi/3)\rho_{\text{obj}}r^3$ – for the same reasons that we listed ρ_{obj} instead of m in the table of variables – and happily ignore the factor of $4\pi/3$. With those choices, the weight is

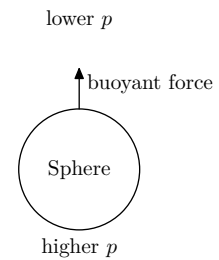
$$F_g \sim \rho_{\text{obj}}r^3g.$$



The figure shows the roadmap updated with this information.

The remaining pieces are drag and buoyancy. Buoyancy is easier, so do it first (the principle of maximal laziness). It is an upward force that results because gravity affects the pressure in a fluid. The pressure increases according to $p = p_0 + \rho_{\text{fl}}gh$, where h is the depth and p_0 is the pressure at zero depth (which can be taken to be at any level in the fluid). The pressure difference between the top and bottom of the object, which are separated by a distance $\sim r$, is $\Delta p \sim \rho_{\text{fl}}gr$. Pressure is force per area, and the pressure difference acts over an area $A \sim r^2$. Therefore the buoyant force created by the pressure difference is

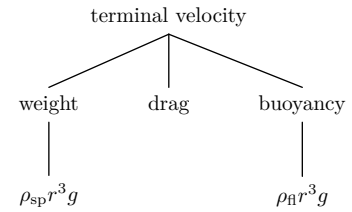
$$F_b \sim A\Delta p \sim \rho_{\text{fl}}r^3g.$$



As a check on this result, Archimedes's principle says that the buoyant force is the 'weight of fluid displaced'. This weight is

$$\underbrace{\rho_{\text{fl}} \frac{4\pi}{3} \pi r^3}_{\text{mass}} g.$$

$\underbrace{\phantom{\rho_{\text{fl}} \frac{4\pi}{3} \pi r^3}}_{\text{volume}}$



Except for the factor of $4\pi/3$, it matches the buoyant force so Archimedes's principle confirms our estimate for F_b . That result updates the roadmap. The main unexplored branch is the drag force, which we solve using dimensional analysis.

7.3.3 Dimensional analysis for the drag force

The weight and buoyancy were solvable without dimensional analysis, but we still need to use dimensional analysis to find the drag force. The purpose of breaking the problem into parts was to simplify this dimensional analysis relative to the brute-force approach in [Section 7.3.1](#). Let's see how the list of variables changes when computing the drag force rather than the terminal velocity. The drag force F_d has to join the list: not a promising beginning when trying to eliminate variables. Worse, the terminal velocity v remains on the list, even though we are no longer computing it, because the drag force depends on the velocity of the object.

However, all is not lost. The drag force has no idea what is inside the sphere. Picture the fluid as a huge computer that implements the laws of fluid dynamics. From the viewpoint of this computer, the parameters v and r are the only relevant attributes of a moving sphere. What lies underneath the surface does not affect the fluid flow: Drag is only skin deep. The computer can determine the flow (if it has tremendous processing power) without knowing the sphere's density ρ_{obj} , which means it vanishes from the list. Progress!

Now consider the characteristics of the fluid. The fluid supercomputer still needs the density and viscosity of the fluid to determine how the pieces of fluid move in response to the object's motion. So ρ_{fl} and ν remain on the list. What about gravity? It causes the object to fall, so it is responsible for the

Var	Dim	What
F_d	MLT^{-2}	drag force
ν	L^2T^{-1}	kinematic viscosity
ρ_{fl}	ML^{-3}	fluid density
r	L	object radius
v	LT^{-1}	terminal velocity

terminal velocity v . However, the fluid supercomputer does not care how the object acquired this velocity; it cares only what the velocity is. So g vanishes from the list. The updated table shows the new, shorter list.

The five variables in the list are composed of three basic dimensions. From the Buckingham Pi theorem (Section 6.6), we expect two dimensionless groups. We find one group by dividing and conquering. The list already includes a velocity (the terminal velocity). If we can concoct another quantity V with dimensions of velocity, then v/V is a dimensionless group. The viscosity ν is almost a velocity. It contains one more power of length than velocity does. Dividing by r eliminates the extra length: $V \equiv \nu/r$. A dimensionless group is then

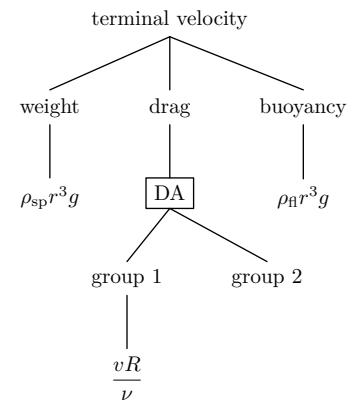
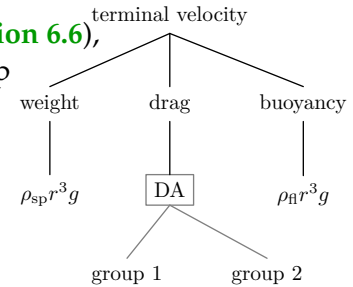
$$G_1 \equiv \frac{v}{V} = \frac{vr}{\nu}.$$

Our knowledge, including this group, is shown in the figure. This group is so important that it has a name, the **Reynolds number**, which is abbreviated Re . It is important because it is a *dimensionless* measure of flow speed. The velocity, because it contains dimensions, cannot distinguish fast from slow flows. For example, 1000 m s^{-1} is slow for a planet, whose speeds are typically tens of kilometers per second, but fast for a pedestrian. When you hear that a quantity is small, fast, large, expensive, or almost any adjective, your first reaction should be to ask, ‘compared to what?’ Such a comparison suggests dividing v by another velocity; then we get a dimensionless quantity that is proportional to v . The result of this division is the Reynolds number.

Low values of Re indicate slow, viscous flow (cold honey oozing out of a jar). High values indicate turbulent flow (a jet flying at 600 mph). The excellent *Life in Moving Fluids* [36] discusses many more dimensionless ratios that arise in fluid mechanics.

The Reynolds number looks lonely in the map. To give it company, find a second dimensionless group. The drag force is absent from the first group so it must live in the second; otherwise we cannot solve for the drag force.

Instead of dreaming up the dimensionless group in one lucky guess, we construct it in steps (divide-and-conquer reasoning). Examine the variables



in the table, dimension by dimension. Only two (F_d and ρ_{fl}) contain mass, so both or neither appear in the group. Because F_d has to appear, ρ_{fl} must also appear. Each variable contains a first power of mass, so the group contains the ratio F_d/ρ_{fl} . A simple choice is

$$G_2 \propto \frac{F_d}{\rho_{fl}}.$$

The dimensions of F_d/ρ_{fl} are L^4T^{-2} , which is the square of L^2T^{-1} . Fortune smiles on us, for L^2T^{-1} are the dimensions of ν . So

$$\frac{F_d}{\rho_{fl}\nu^2}$$

is a dimensionless group.

This choice, although valid, has a defect: It contains ν , which already belongs to the first group (the Reynolds number). Of all the variables in the problem, ν is the one most likely to be found irrelevant based on a physical argument (as will happen in [Section 7.3.7](#), when we specialize to high-speed flow. If ν appears in two groups, eliminating it requires recombining the two groups into one that does not contain ν . However, if ν appears in only one group, then eliminating it is simple: eliminate that group. Simpler mathematics – eliminating a group rather than remixing two groups to get one group – requires simpler physical reasoning. Therefore, isolate ν in one group if possible.

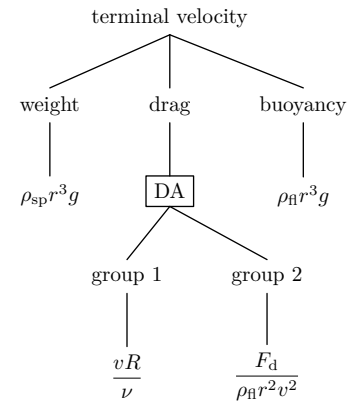
To remove ν from the proposed group $F_d/\rho_{fl}\nu^2$ notice that the product of two dimensionless groups is also dimensionless. The first group contains ν^{-1} and the proposed group contains ν^{-2} , so the ratio

$$\frac{\text{group proposed}}{(\text{first group})^2} = \frac{F_d}{\rho_{fl}r^2\nu^2}$$

is not only dimensionless but it also does not contain ν . So the analysis will be easy to modify when we try to eliminate ν . With this revised second group, our knowledge is now shown in this figure:

This group, unlike the the proposal $F_d/\rho_{fl}\nu^2$, has a plausible physical interpretation. Imagine that the sphere travels a distance l , and use l to multiply the group by unity:

$$\underbrace{\frac{F_d}{\rho_{fl}r^2\nu^2}}_{\text{group 1}} \times \underbrace{\frac{l}{l}}_1 = \frac{F_d l}{\rho_{fl} l r^2 \nu^2}.$$



The numerator is the work done against the drag force over the distance l . The denominator is also an energy. To interpret it, examine its parts (divide and conquer). The product lr^2 is, except for a dimensionless constant, the volume of fluid swept out by the object. So $\rho_{fl}lr^2$ is, except for a constant, the mass of fluid shoved aside by the object. The object moves fluid with a velocity comparable to v , so it imparts to the fluid a kinetic energy

$$E_K \sim \rho_{fl}lr^2v^2.$$

Thus the ratio, and hence the group, has the following interpretation:

$$\frac{\text{work done against drag}}{\text{kinetic energy imparted to the fluid}}.$$

In highly dissipative flows, when energy is burned directly up by viscosity, the numerator is much larger than the denominator, so this ratio (which will turn out to measure drag) is much greater than 1. In highly streamlined flows (a jet wing), the the work done against drag is small because the fluid returns most of the imparted kinetic energy to the object. So in the ratio, the numerator will be small compared to the denominator.

To solve for F_d , which is contained in G_2 , use the form $G_2 = f(G_1)$, which becomes

$$\frac{F_d}{\rho_{fl}r^2v^2} = f\left(\frac{vr}{\nu}\right).$$

The drag force is then

$$F_d = \rho_{fl}r^2v^2 f\left(\frac{vr}{\nu}\right).$$

The function f is a dimensionless function: Its argument is dimensionless and it returns a dimensionless number. It is also a universal function. The same f applies to spheres of any size, in a fluid of any viscosity or density! Although f depends on r , ρ_{fl} , ν , and v , it depends on them only through one combination, the Reynolds number. A function of one variable is easier to study than is a function of four variables:

A good table of functions of one variable may require a page; that of a function of two variables a volume; that of a function of three variables a bookcase; and that of a function of four variables a library.

—Harold Jeffreys [25, p. 82]

Dimensional analysis cannot tell us the form of f . To learn its form, we specialize to two special cases:

1. viscous, low-speed flow ($Re \ll 1$), the subject of [Section 7.3.4](#); and
2. turbulent, high-speed flow ($Re \gg 1$), the subject of [Section 7.3.7](#).

7.3.4 Viscous limit

As an example of the low-speed limit, consider a marble falling in vegetable oil or glycerin. You may wonder how often marbles fall in oil, and why we bother with this example. The short answer to the first question is ‘not often’. However, the same physics that determines the fall of marbles in oil also determines, for example, the behavior of fog droplets in air, of bacteria swimming in water [26], or of oil drops in the Millikan oil-drop experiment. The marble problem not only illustrates the physical principles, but also we can check our results with a home experiment.

In slow, viscous flows, the drag force comes directly from – surprise! – viscous forces. These forces are proportional to viscosity because viscosity is the constant of proportionality in the definition of the viscous force. Therefore

$$F_d \propto \nu.$$

The viscosity appears exactly once in the drag result, repeated here:

$$F_d = \rho_{fl} r^2 v^2 f\left(\frac{\nu r}{v}\right).$$

To flip ν into the numerator and make $F_d \propto \nu$, the function f must have the form $f(x) \sim 1/x$. With this $f(x)$ the result is

$$F_d \sim \rho_{fl} r^2 v^2 \frac{\nu}{\nu r} = \rho_{fl} \nu v.$$

Dimensional analysis alone is insufficient to compute the missing magic dimensionless constant. A fluid mechanician must do a messy and difficult calculation. Her burden is light now that we have worked out the solution except for this one constant. The British mathematician Stokes, the first to derive its value, found that

$$F_d = 6\pi \rho_{fl} \nu r v.$$

In honor of Stokes, this result is called Stokes drag.

Let’s sanity check the result. Large or fast marbles should feel a lot of drag, so r and v should be in the numerator. Viscous fluids should produce a

lot of drag, so ν should be the numerator. The proposed drag force passes these tests. The correct location of the density – in the numerator or denominator – is hard to judge.

You can make an educated judgment by studying the Navier–Stokes equations. In those equations, when ν is ‘small’ (small compared to what?) then the $(\mathbf{v} \cdot \nabla) \mathbf{v}$ term, which contains two powers of ν , becomes tiny compared to the viscous term $\nu \nabla^2 \mathbf{v}$, which contains only one power of ν . The second-order term arises from the inertia of the fluid, so this term’s being small says that the oozing marble does not experience inertial effects. So perhaps ρ_{fl} , which represents the inertia of the fluid, should not appear in the Stokes drag. On the other hand, viscous forces are proportional to the *dynamic* viscosity $\eta = \rho_{fl} \nu$, so ρ_{fl} should appear even if inertia is unimportant. The Stokes drag passes this test. Using the dynamic instead of kinematic viscosity, the Stokes drag is

$$F_d = 6\pi\eta\nu r,$$

often a convenient form because many tables list η rather than ν .

This factor of 6π comes from doing honest calculations. Here, it comes from solving the Navier–Stokes equations. In this book we wish to teach you how not to suffer, so we do not solve such equations. We usually quote the factor from honest calculation to show you how accurate (or sloppy) the approximations are. The factor is often near unity, although not in this case where it is roughly 20! In fancy talk, it is usually ‘of order unity’. Such a number suits our neural hardware: It is easy to remember and to use. Knowing the approximate derivation and remembering this one number, you reconstruct the exact result without solving difficult equations.

Now use the Stokes drag to estimate the terminal velocity in the special case of low Reynolds number.

7.3.5 Terminal velocity for low Reynolds number

Chapter 7. Special cases

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Having assembled all the pieces in the roadmap, we now return to the original problem of finding the terminal velocity. Since no net force acts on the marble (the definition of terminal velocity), the drag force plus the buoyant force equals the weight:

$$\underbrace{\nu \rho_{fl} \nu r}_{F_d} + \underbrace{\rho_{fl} g r^3}_{F_b} \sim \underbrace{\rho_{obj} g r^3}_{F_g}.$$

After rearranging:

$$\nu \rho_{fl} \nu r \sim (\rho_{obj} - \rho_{fl}) g r^3.$$

The terminal velocity is then

$$\nu \sim \frac{g r^2}{\nu} \left(\frac{\rho_{obj}}{\rho_{fl}} - 1 \right).$$

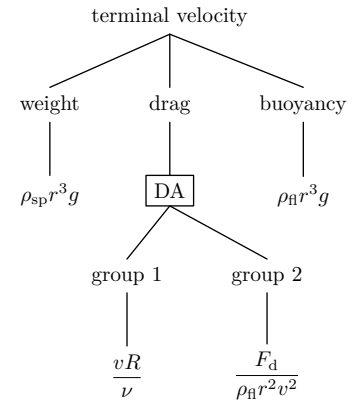
In terms of the dynamic viscosity η , it is

$$\nu \sim \frac{g r^2}{\eta} (\rho_{obj} - \rho_{fl}).$$

This version, instead of having the dimensionless factor $\rho_{obj}/\rho_{fl} - 1$ that appears in the version with kinematic viscosity, has a dimensional $\rho_{obj} - \rho_{fl}$ factor. Although it is less aesthetic, it is often more convenient because tables often list dynamic viscosity η rather than kinematic viscosity ν .

We can increase our confidence in this expression by checking whether the correct variables are upstairs (a picturesque way to say ‘in the numerator’) and downstairs (in the denominator). Denser marbles should fall faster than less dense marbles, so ρ_{obj} should live upstairs. Gravity accelerates marbles, so g should live upstairs. Viscosity slows marbles, so ν should live downstairs. The terminal velocity passes these tests. We therefore have more confidence in our result, although the tests did not check the location of r or any exponents: For example, should ν appear as ν^2 ? Who knows, but if viscosity matters, it mostly appears as a square root or as a first power.

To check r , imagine a large marble. It will experience a lot of drag and fall slowly, so r should appear downstairs. However, large marbles are also heavy and fall rapidly, which suggests that r should appear upstairs. Which effect wins is not obvious, although after you have experience with these problems, you can make an educated guess: weight scales as r^3 , a rapidly rising function r , whereas drag is probably proportional to a lower



power of r . Weight usually wins such contents, as it does here, leaving r upstairs. So the terminal velocity also passes the r test.

Let's look at the dimensionless ratio in parentheses: $\rho_{\text{obj}}/\rho_{\text{fl}} - 1$. Without buoyancy the -1 disappears, and the terminal velocity would be

$$v \propto g \frac{\rho_{\text{obj}}}{\rho_{\text{fl}}}.$$

We retain the g in the proportionality for the following reason: The true solution returns if we replace g by an effective gravity g' where

$$g' \equiv g \left(1 - \frac{\rho_{\text{fl}}}{\rho_{\text{obj}}} \right).$$

So, one way to incorporate the effect of the buoyant force is to solve the problem without buoyancy but with the reduced g .

Check this replacement in two limiting cases: $\rho_{\text{fl}} = 0$ and $\rho_{\text{fl}} = \rho_{\text{obj}}$. When $\rho_{\text{obj}} = \rho_{\text{fl}}$ gravity vanishes: People, whose density is close to the density of water, barely float in swimming pools. Then g' should be zero. When $\rho_{\text{fl}} = 0$, buoyancy vanishes and gravity retains its full effect. So g' should equal g . The effective gravity definition satisfies both tests. Between these two limits, the effective g should vary linearly with ρ_{fl} because buoyancy and weight superpose linearly in their effect on the object. The effective g passes this test as well.

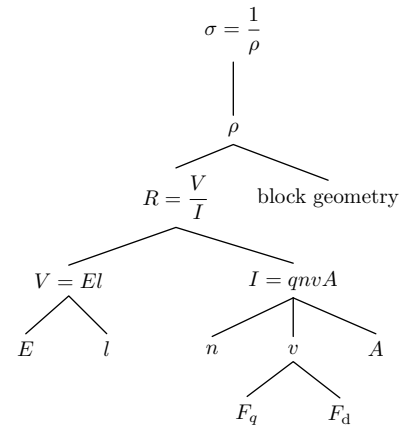
Another test is to imagine $\rho_{\text{fl}} > \rho_{\text{obj}}$. Then the relation correctly predicts that g' is negative: helium balloons rise. This alternative to using buoyancy explicitly is often useful. If, for example you forget to include buoyancy (which happened in the first draft of this chapter), you can correct the results later by replacing g with the g' .

If we carry forward the constants of proportionality, starting with the magic 6π in the Stokes drag and including the $4\pi/3$ that belongs in the weight, we find

$$v \sim \frac{2}{9} \frac{gr^2}{\nu} \left(\frac{\rho_{\text{obj}}}{\rho_{\text{fl}}} - 1 \right).$$

7.3.6 Conductivity of seawater

As an application of Stokes drag and a rare example of a realistic situation with low Reynolds numbers, let's estimate the electrical conductivity of seawater. Solving this problem is hopeless without breaking it into pieces. Conductivity σ is the reciprocal of resistivity ρ . (Apologies for the convention that overloads the density symbol with yet another meaning.) Resistivity, as its name suggests, is related to resistance R . Why have both ρ and R ? Resistance is a useful measure for a particular wire, but not for wires in general because it depends on the diameter and cross-sectional area of the wire. It is not an intensive quantity. Before examining the relationship between resistivity and resistance, let's finish sketching the solution tree, leaving ρ as depending on R plus geometry. We can find R by placing a voltage V across a block of seawater and measuring the current I ; then $R = V/I$.



To find V or I we need a physical model. First, why does seawater conduct at all? Conduction requires the transport of charge, which is produced by an electric field. Seawater is mostly water and table salt (NaCl). The ions that arise from dissolving salt can transport charge. The resulting current is

$$I = qnvA,$$

where A is the cross-sectional area of the block, q is the ion charge, n is the ion concentration, and v is its terminal speed.

To understand, and be able to rederive this formula, first check its dimensions. Current is charge per time. Is the right side also charge per time? Yes: q takes care of the charge; and vA has dimensions of L^3T^{-1} so nvA , which has dimensions of T^{-1} , takes care of the time.

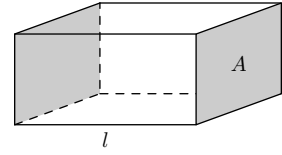
As a second check, watch a cross-section of the block for a time Δt . How much charge flows in that time? The charges move at speed v , so all charges in block of width $v\Delta t$ and area A cross the cross-section. This block has volume $vA\Delta t$. The ion concentration is n , so the block contains $nvA\Delta t$ charges. If each ion has charge q , then the total charge on the ions is $Q = qnvA\Delta t$. It took a time Δt for this charge to flow, so the current is $I = Q/\Delta t = qnvA$. The terminal speed v depends on the applied force F_q and on the drag force F_d , just as for the falling marble but with an electrical force instead of a gravitational force. The result of this subdividing is the preceding map.

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7.3. Drag

Now let's find expressions for the unknown nodes. Only three remain: ρ , v , and n . The figure illustrates the relation between ρ and R :

$$\rho = \frac{RA}{l}.$$



To find v we follow the same procedure as for the marble. The applied force is $F_q = qE$, where q is the ion charge and E is the electric field. The electric field produced by the voltage V is $E = V/l$, where l is the length of the block, so

$$F_q = \frac{qV}{l},$$

an expression in terms only of known quantities. The drag is Stokes drag. Equating this drag to the applied force gives the terminal velocity v in terms of known quantities:

$$v \sim \frac{qV}{6\pi\eta lr},$$

where r is the radius of the ion.

Only the number density n remains unknown. We estimate it after getting a symbolic result for σ , which you can do by climbing up the solution tree. First, find the current in terms of the terminal velocity:

$$I = qnvA \sim \frac{q^2 nAV}{6\pi\eta lr}.$$

Use the current to find the resistance:

$$R \sim \frac{V}{I} \sim \frac{6\pi\eta lr}{q^2 nA}.$$

The voltage V has vanished, which is encouraging: In most circuits the conductivity (and resistance) is independent of voltage. Use the resistance to find the resistivity:

$$\rho = R \frac{A}{l} \sim \frac{6\pi\eta r}{q^2 n}.$$

The expression simplifies as we rise up the tree: The geometric parameters l and A have also vanished, which is also encouraging: The purpose of evaluating resistivity rather than resistance is that resistivity is independent of geometry.

Use resistivity to find conductivity:

$$\sigma = \frac{1}{\rho} \sim \frac{q^2 n}{6\pi\eta r}.$$

Here q is the electron charge e or its negative, depending on whether a sodium or a chloride ion is the charge carrier, so

$$\sigma = \frac{1}{\rho} \sim \frac{e^2 n}{6\pi\eta r}.$$

To find σ still requires the ion concentration n , which we can find from the concentration of salt in seawater. This value I estimate with a kitchen-sink experiment: Add table salt to a glass of water until it tastes as salty as seawater. I just tried it. In a glass of water, I found that a teaspoon of salt tastes very salty, like drinking seawater. A glass of water may have a volume of 0.3ℓ or a mass of 300 g . A flat teaspoon of salt has a volume of about 5 ml . For those who live in metric countries, a teaspoon is an archaic measure used in Britain and especially the United States, which has no nearby metric country to which it pays attention. A teaspoon is about 4 cm long by 2 cm wide by 1 cm thick at its deepest point; let's assume 0.5 cm on average. Its volume is therefore

$$\text{teaspoon} \sim 4 \text{ cm} \times 2 \text{ cm} \times 0.5 \text{ cm} \sim 4 \text{ cm}^3.$$

The density of salt is maybe twice the density of water, so a flat teaspoon has a mass of $\sim 10 \text{ g}$. The mass fraction of salt in seawater is, in this experiment, roughly $1/30$. The true value is remarkably close: 0.035 . A mole of salt, which provides two charges per NaCl 'molecule', has a mass of 60 g , so

$$\begin{aligned} n &\sim \frac{1}{30} \times \underbrace{1 \text{ g cm}^{-3}}_{\rho_{\text{water}}} \times \frac{2 \text{ charges}}{\text{molecule}} \times \frac{6 \cdot 10^{23} \text{ molecules mol}^{-1}}{60 \text{ g mol}^{-1}} \\ &\sim 7 \cdot 10^{20} \text{ charges cm}^{-3}. \end{aligned}$$

With n evaluated, the only remaining mysteries in the conductivity

$$\sigma = \frac{1}{\rho} \sim \frac{q^2 n}{6\pi\eta r}$$

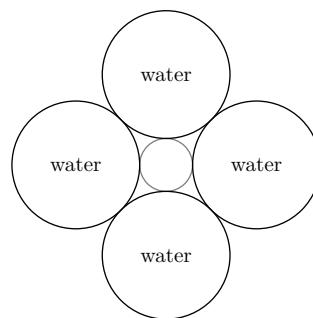
are the ion radius r and the dynamic viscosity η .

Do the easy part first. The dynamic viscosity is

$$\eta = \rho_{\text{water}} \nu \sim 10^3 \text{ kg m}^{-3} \times 10^{-6} \text{ m}^2 \text{ s}^{-1} = 10^{-3} \text{ kg m}^{-1} \text{ s}^{-1}.$$

Here I switched to SI (mks) units. Although most calculations are easier in cgs units – also known as God’s units – than they are in SI units, the one exception is electromagnetism, which is represented by the e^2 in the conductivity. Electromagnetism is conceptually easier in cgs units – which needs no ghastly μ_0 or $4\pi\epsilon_0$, for example – than it is in SI units. However, the cgs unit of charge, the electrostatic unit, is unfamiliar. So, for numerical calculations, use SI units.

The final quantity required is the ion radius. A positive ion (sodium) attracts an oxygen end of a water molecule; a negative ion (chloride) attracts the hydrogen end of a water molecule. Either way, the ion, being charged, is surrounded by one or maybe more layers of water molecules. As it moves, it drags some of this baggage with it. So rather than use the bare ion radius you should use a larger radius to include this shell. But how thick is the shell? As an educated guess, assume that the shell includes one layer of water molecules, each with a radius of 1.5 Å. So for the ion plus shell, $r \sim 2$ Å.



With these numbers, the conductivity becomes:

$$\sigma \sim \frac{\overbrace{(1.6 \cdot 10^{-19} \text{ C})^2}^{e^2} \times \overbrace{7 \cdot 10^{26} \text{ m}^{-3}}^n}{\underbrace{6 \times 3}_{6\pi} \times \underbrace{10^{-3} \text{ kg m}^{-1} \text{ s}^{-1}}_{\eta} \times \underbrace{2 \cdot 10^{-10} \text{ m}}_r}.$$

You can do the computation mentally: *Take out the big part, apply the principle of maximal laziness, and divide and conquer* by first counting the powers of ten (shown in red) and then worrying about the small factors. Then *divide and conquer* again by counting the top and bottom contributions separately. The top contributes -12 powers of ten: -38 from e^2 and +26 from n . The bottom contributes -13 powers of ten: -3 from η and -10 from r . The division produces one power of ten.

Now account for the remaining small factors:

$$\frac{1.6^2 \times 7}{6 \times 3 \times 2}.$$

Slightly overestimate the answer by pretending that the 1.6^2 on top cancels the 3 on the bottom. Slightly underestimate the answer – and maybe compensate for the overestimate – by pretending that the 7 on top cancels the

6 on the bottom. After these lies, only $1/2$ remains. Multiplying it by the sole power of ten gives

$$\sigma \sim 5 \Omega^{-1} \text{ m}^{-1}.$$

Using a calculator to do the arithmetic gives $4.977 \dots \Omega^{-1} \text{ m}^{-1}$, which is extremely close to the result from mental calculation.

The estimated resistivity is

$$\rho \sim \sigma^{-1} \sim 0.2 \Omega \text{ m} = 20 \Omega \text{ cm},$$

where we converted to the conventional although not fully SI units of $\Omega \text{ cm}$. A typical experimental value for seawater at $T = 15^\circ \text{C}$ is $23.3 \Omega \text{ cm}$ (from [p. 14-15]), absurdly close to the estimate!

Probably the most significant error is the radius of the ion-plus-water combination that is doing the charge transport. Perhaps r should be greater than $2 \text{ \AA}$, especially for a sodium ion, which is smaller than chloride; it therefore has a higher electric field at its surface and grabs water molecules more strongly than chloride does. In spite of such uncertainties, the continuum approximation produced more accurate results than it ought to.

At the length scale of a sodium ion, water looks like a collection of spongy boulders more than it looks like a continuum. Yet Stokes drag worked. It works because the important length scale is not the size of water molecules, but rather their mean free path between collisions. Molecules in a liquid are packed to the point of contact, so the mean free path is much shorter than a molecular (or even ionic) radius, especially compared to an ion with its shell of water.

The moral of this example, besides illustrating Stokes drag, is to have courage. Approximate first and ask questions later. Maybe the approximations are correct for reasons that you do not suspect when you start solving a problem. If you agonize over each approximation, you will never start a calculation, and then you will not find out that many approximations would have been fine. . .if only you had had the courage to make them.

7.3.7 Turbulent limit

We now compute drag in the other flow extreme: high-speed, or turbulent, flow. The example will be to compute the terminal speed of a raindrop. These results apply to most flows. For example, when a child rises from a chair, the airflow around her is high-speed flow, as you can check by

computing the Reynolds number. Say that the child is 0.2 m wide, and that she rises with velocity 0.5 m s^{-1} . Then

$$\text{Re} \sim \frac{vr}{\nu_{\text{air}}} \sim \frac{0.5 \text{ m s}^{-1} \times 0.2 \text{ m}}{2 \cdot 10^{-5} \text{ m}^2 \text{ s}^{-1}} \sim 5000.$$

Here viscosity of air is closer to

$$\nu_{\text{air}} \approx 1.5 \cdot 10^{-5} \text{ m}^2 \text{ s}^{-1},$$

than to $2 \cdot 10^{-5} \text{ m}^2 \text{ s}^{-1}$, but $2 \cdot 10^{-5} \text{ m}^2 \text{ s}^{-1}$ easily combines with the 0.2 m in the numerator to allow us to do the calculation mentally. Using either value for the viscosity, the Reynolds number is much larger than unity, so the flow is turbulent. Larger objects, such as planes, trains, and automobiles, create turbulence even when they travel even more slowly than the child. In short, most fluid flow around us is turbulent flow.

To begin the analysis, we assume that a raindrop is a sphere. It is a convenient lie that allows us to reuse the general results of [Section 7.3.3](#) and specialize to high-speed flow. At high speeds (more precisely, at high Reynolds number) the flow is turbulent. Viscosity – which affects only slow flows but does not directly influence the shearing and whirling of turbulent flows – becomes irrelevant. Let's see how much we can understand about turbulent drag knowing only that turbulent drag is nearly independent of viscosity.

Turbulence is perhaps the main unsolved problem in classical physics. However, you can still understand a lot about drag using dimensional analysis plus a bit of physical reasoning; we do not need a full understanding of turbulence. The world is messy: Do not wait for a full understanding before you analyze or estimate.

In the roadmap for low Reynolds number, the viscosity appears only in the first group. Because turbulent drag is independent of the viscosity, the viscosity disappears from the results and therefore so does that group. This argument is glib. More precisely, remove ν from the list of variables and search again for dimensionless groups. The remaining four variables, shown in the table, result in one dimensionless group, which is the second group from the old roadmap.

Var	Dim	What
F_d	MLT^{-2}	drag force
ρ_{fl}	ML^{-3}	fluid density
r	L	object radius
v	LT^{-1}	terminal velocity

So the Reynolds number, which was the first group, has disappeared from the analysis. But why is drag at high speeds independent of Reynolds number? Equivalently, why can we remove ν from the list of variables and still

get the correct form for the drag force? The answer is not obvious. The explanation of the Reynolds number as a ratio of two speeds v and V provides a partial answer. A natural length in this problem is r ; we can use r to transform v and V into times:

$$\tau_v \equiv \frac{r}{v},$$

$$\tau_V \equiv \frac{r}{V} \sim \frac{r^2}{\nu}.$$

Note that $Re \equiv \tau_V/\tau_v$. The quantity τ_v is the time that fluid takes to travel around the sphere (apart from constants). Kinematic viscosity is ν/ρ , but its most important interpretation is as the diffusion coefficient for momentum. So the time for momentum to diffuse a distance x is

$$\tau \sim \frac{x^2}{\nu}.$$

This result depends on the mathematics of random walks; you can increase your confidence in it here, without understanding the theory of random walks, by checking that it has valid dimensions. And it has: Each side is a time.

So τ_V is the time that momentum takes to diffuse around an object of size r , such as the falling sphere in this problem. If $\tau_V \ll \tau_v$ – in which case $Re \ll 1$ – then momentum diffuses before fluid travels around the sphere. Momentum diffusion equalizes velocities, if it has time, which it does have in this low-Reynolds-number limit. Momentum diffusion therefore prevents flow at the front from being radically different from the flow at the back, and thereby squelches any turbulence. In the other limit, when $\tau_V \gg \tau_v$ or $Re \gg 1$ – momentum diffusion is outraced by fluid flow, so the fluid is free to shred itself into a turbulent mess. Once the viscosity is low enough to allow turbulence, its value does not affect the drag, which is why we can ignore it for $Re \gg 1$. Here $Re \gg 1$ means ‘large enough so that turbulence sets in’, which happens around $Re \sim 1000$. A more complete story, which we discuss as part of boundary layers in ??, slightly corrects this approximation. However, it is close enough for our purposes here.

Here the important point is that the viscosity vanishes from the analysis and so does group 1. Once it disappears, the dimensionless group that remains is

$$G_2 = \frac{F_d}{\rho_{fl} r^2 v^2}.$$

Because it is the only group, the solution is

$$G_2 = \text{dimensionless constant},$$

or

$$F_d \sim \rho_{fl} r^2 v^2.$$

This drag is for a sphere. What about other shapes, which are characterized by more parameters than a sphere is? So that the drag force generalizes to more complex shapes, we express it using the cross-sectional area of the object. Here $A = \pi r^2$, so

$$F_d \sim \rho_{fl} A v^2.$$

This conventional choice has a physical basis. As an object moves, the mass of fluid that it displaces is proportional to its cross-sectional area:

$$m_{fl} = \rho_{fl} A h.$$

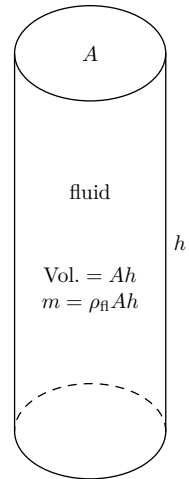
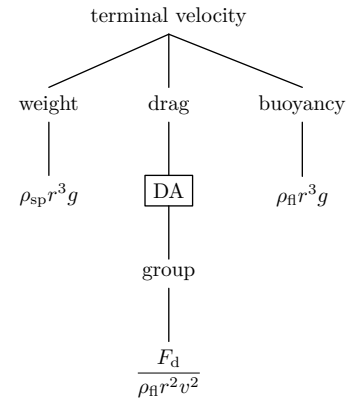
The fluid is given a speed comparable to v , so the fluid's kinetic energy is

$$E_K \sim \frac{1}{2} m_{fl} v^2 \sim \frac{1}{2} \rho_{fl} A h v^2.$$

If all this kinetic energy is dissipated by drag, then the drag force is E_K/h or

$$F_d \sim \frac{1}{2} \rho_{fl} A v^2.$$

In this form with the factor of $1/2$, the constant of proportionality is the **drag coefficient** c_d .



Like its close cousin f from the dimensionless drag force, the drag coefficient is a dimensionless measure of the drag force. It depends on the shape of the object – on how streamlined it is. The table lists c_d for various shapes (at high Reynolds number). The drag coefficient, being proportional to the function $f(Re)$ in the general solution, also depends on the Reynolds number. However, using the reasoning that the flow at high Reynolds number is independent of viscosity, the drag coefficient should also be independent of Reynolds number. Using the drag coefficient instead of f (which implies using cross-sectional area instead of r^2), the turbulent drag force becomes

Object	c_d
Sphere	0.5
Cylinder	1.0
Flat plate	2.0
Car	0.4

$$F_d = \frac{1}{2} c_d \rho_{fl} v^2 A.$$

So we have an expression for the turbulent drag force. The weight and buoyant forces are the same as in the viscous limit. So we just need to redo the analysis of the viscous limit but with the new drag force. Because the weight and buoyant forces contain r^3 , we return to using r^2 instead of A in the drag force. With these results, the terminal velocity v is given by

$$\underbrace{\rho_{fl} r^2 v^2}_{F_d} \sim \underbrace{g(\rho_{obj} - \rho_{fl}) r^3}_{F_g - F_b},$$

so

$$v \sim \sqrt{gr \left(\frac{\rho_{obj}}{\rho_{fl}} - 1 \right)}.$$

Pause to sanity check this result: Are the right variables upstairs and downstairs? We consider each variable in turn.

- ρ_{fl} : The terminal velocity is smaller in a denser fluid (try running in a swimming pool), so ρ_{fl} should be in the denominator.
- g : Imagine a person falling on a planet that has a gravitational force stronger than that of the earth. Gravity partially determines atmospheric pressure and density. Holding the atmospheric density constant while increasing gravity might be impossible in real life, but we can do it easily in a thought experiment. The drag force then does not depend on g , so gravity increases the terminal speed without opposition from the drag force: g should be upstairs.
- ρ_{obj} : Imagine a raindrop made of (very) heavy water. Relative to a standard raindrop, the gravitational force increases while the drag force

remains constant, as shown using the fluid-is-a-computer argument in ??sec:drag-force-DA. So ρ_{obj} should be upstairs.

- r : To determine where the radius lives requires a more subtle argument. Increasing r increases both the gravitational and drag forces. The gravitational force increases as r^3 whereas the drag force increases only as r^2 . So, for larger raindrops, their greater weight increases v more than their greater drag decreases v . Therefore r should be live upstairs.
- ν : At high Reynolds number viscosity does not affect drag, at least not in our approximation. So ν should not appear anywhere.

The terminal velocity passes all tests.

Now we can compute the terminal velocity. The splash spots on the sidewalk made by raindrops in a recent rain have $r \sim 0.3 \text{ cm}$. Since rain is water, its density is $\rho_{\text{obj}} \sim 1 \text{ g cm}^{-3}$. The density of air is $\rho_{\text{fl}} \sim 1 \text{ kg m}^{-3}$, so $\rho_{\text{fl}} \ll \rho_{\text{obj}}$: Buoyancy is therefore not an important effect, and we can replace $\rho_{\text{obj}}/\rho_{\text{fl}} - 1$ by $\rho_{\text{obj}}/\rho_{\text{fl}}$. With this simplification and the estimated numbers, the terminal velocity is:

$$v \sim \left(\underbrace{1000 \text{ cm s}^{-2}}_g \times \underbrace{0.3 \text{ cm}}_r \times \underbrace{\frac{1 \text{ g cm}^{-3}}{10^{-3} \text{ g cm}^{-3}}}_{\frac{\rho_{\text{obj}}}{\rho_{\text{fl}}}} \right)^{1/2} \sim 5 \text{ m s}^{-1},$$

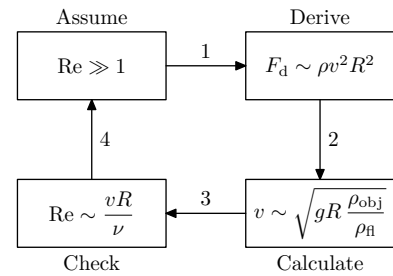
or 10 mph.

This calculation assumed that $\text{Re} \gg 1$. Check that assumption! You need not calculate Re from scratch; rather, scale it relative to a previous results. As we worked out earlier, a child ($r \sim 0.2 \text{ m}$) rising from her chair ($v \sim 0.5 \text{ m s}^{-1}$) creates a turbulent flow with $\text{Re} \sim 5000$. The flow created by the raindrop is faster by a factor of 10, but the raindrop is smaller by a factor of roughly 100. Scaling the Reynolds number for the child gives

$$\text{Re} \sim \underbrace{\text{Re}_{\text{child}}}_{5000} \times \underbrace{\left(\frac{v_{\text{drop}}}{v_{\text{child}}} \right)}_{10} \times \underbrace{\left(\frac{r_{\text{drop}}}{r_{\text{child}}} \right)}_{0.01} \sim 500.$$

This Reynolds number is also much larger than 1, so the flow produced by the raindrop is turbulent, which vindicates the initial assumption.

Now that we have found the terminal velocity, let's extract the pattern of the solution. The order that we followed was *assume, derive, calculate, then check*. This order is more fruitful than is the simpler order of *derive then calculate*. Without knowing whether the flow is fast or slow, we cannot derive a closed-form expression for F_d ; such a derivation is probably beyond present understanding of fluids and turbulence. Blocked by this mathematical Everest, we would remain trapped in the *derive* box. We would never determine F_d , so we would never realize that the Reynolds number is large (the *assume* box); however, only this assumption makes it possible to eliminate v and thereby to estimate F_d . The moral: Assume early and often!



7.3.8 Combining solutions from the two limits

You know know the drag force in two extreme cases, viscous and turbulent drag. The results are repeated here:

$$F_d = \begin{cases} 6\pi\rho_{fl}\nu vr & \text{(viscous),} \\ \frac{1}{2}c_d\rho_{fl}Av^2 & \text{(turbulent).} \end{cases}$$

Let's compare and combine them by making the viscous form look like the turbulent form. Compared to the turbulent form, the viscous form lacks one power of r and one power of v but has an extra power of ν . A combination of variables with a similar property is the Reynolds number rv/ν . So multiply the viscous drag by a useful form of unity:

$$F_d = \underbrace{\left(\frac{rv/\nu}{Re}\right)}_1 \times \underbrace{6\pi\rho_{fl}\nu\rho_{fl}\nu r}_{F_d} = \frac{1}{Re} 6\pi\rho_{fl}\nu^2 r^2 \quad \text{(viscous).}$$

This form, except for the 6π and the r^2 , resembles the turbulent drag. Fortunately $A = \pi r^2$ so

$$F_d = \frac{6}{Re} \rho_{fl}\nu^2 A \quad \text{(viscous),}$$

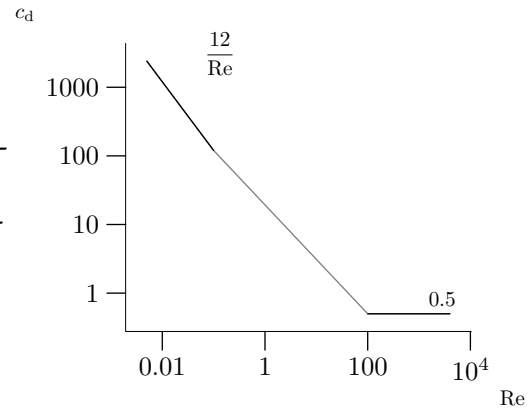
With

$$c_d = \frac{12}{Re} \quad \text{(viscous),}$$

the turbulent drag and this rewritten viscous drag for a sphere have the same form:

$$F_d = \frac{1}{2} \rho_{fl} A v^2 \times \begin{cases} \frac{12}{Re} & (Re \ll 1), \\ 0.5 & (Re \gg 1). \end{cases}$$

At high Reynolds number the drag coefficient remains constant. For a sphere, that constant is $c_d \sim 1/2$. If the low-Reynolds-number approximation for c_d is valid at sufficiently high Reynolds numbers, then c_d would cross $1/2$ near $Re \sim 24$, where presumably the high-Reynolds-number approximation takes over. The crossing point is a reasonable estimate for the transition between low- and high-speed flow. Experiment or massive simulation are the only ways to get a more accurate result. Experimental data place the crossover near $Re \sim 5$, at which point $c_d \sim 2$. Why can't you calculate this value analytically? If a dimensionless variable, such as the Reynolds number, is close to unity, calculations become difficult. Approximations that depend on a quantity being either huge or tiny are no longer valid. When all terms in an equation are roughly of the same magnitude, you cannot get rid of any term without making large errors. To get results in these situations, you have to do honest work: You must do experiments or solve the Navier–Stokes equations numerically.



More on special cases

Summary:

Further reading:

Problem 7.1 Integrals

Use special cases of a to choose the correct formula for each integral.

- a. $\int_{-\infty}^{\infty} e^{-ax^2} dx$
 (1.) $\sqrt{\pi a}$ (2.) $\sqrt{\pi/a}$
- b. $\int_{-\infty}^{\infty} \frac{1}{a^2 + x^2} dx$

(1.) πa (2.) π/a (3.) $\sqrt{\pi}a$ (4.) $\sqrt{\pi}/a$

Problem 7.2 Debugging

Use special (i.e. easy) cases of n to decide which of these two C functions correctly computes the sum of the first n odd numbers:

```
int sum_of_odds (int n) {
    int i, total = 0;
    for (i=1; i<=2*n+1; i+=2)
        total += i;
    return total;
}
```

or

```
int sum_of_odds (int n) {
    int i, total = 0;
    for (i=1; i<=2*n-1; i+=2)
        total += i;
    return total;
}
```

Problem 7.3 Reynolds numbers

Estimate the Reynolds number for:

- a falling raindrop;
- a flying mosquito;

Problem 7.4 Drag at low Reynolds number

At low Reynolds number, the drag on a sphere is

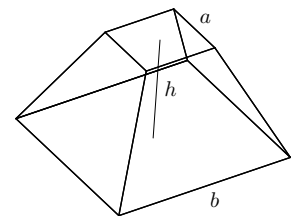
$$F = 6\pi\rho\nu vr.$$

What is the drag coefficient c_d as a function of Reynolds number Re ?

Problem 7.5 Truncated pyramid

In this problem you use special cases to find the volume of a truncated pyramid. It has a square base with side b , a square top with side a , and height h . So, use special cases of a and b to evaluate these candidates for the volume:

- $\frac{1}{3}hb^2$
- $\frac{1}{3}ha^2$



c. $\frac{1}{3}h(a^2 + b^2)$

d. $\frac{1}{2}h(a^2 + b^2)$

Which if any of these formulas pass all your special-cases tests? If no formula passes all tests, invent a formula that does. If you are stuck, find the volume by integration!

Problem 7.6 Fog

- Estimate the terminal speed of fog droplets (radius $\sim 10\ \mu\text{m}$). Use either the low- or high-Reynolds-number limit for the drag force, whichever you guess is the more likely to be valid.
- Use the speed to estimate the Reynolds number and check that you used the correct limit for the drag force. If not, try the other limit!

It is much less than 1, so the original assumption of low-Reynolds-number flow is okay.

- Fog is a low-lying cloud. How long would fog droplets take to fall 1 km (the height of a typical cloud)? What is the everyday effect of this settling time?

Problem 7.7 Tube flow

In this problem you study fluid flow through a narrow tube. The quantity to predict is Q , the volume flow rate (volume per time). This rate depends on:

l	the length of the tube
Δp	the pressure difference between the tube ends
r	the radius of the tube
ρ	the density of the fluid
ν	the kinematic viscosity of the fluid

- Find three independent dimensionless groups G_1 , G_2 , and G_3 from these six variables. *Hint 1:* One physically reasonable group is $G_2 = r/l$. *Hint 2:* Put Q in G_1 only! Then write the general form

$$G_1 = f(G_2, G_3).$$

[There are lots of choices for G_1 and G_3 .]

- Now imagine that the tube is very long and thin ($l \gg r$) and that the radius or flow speed are small enough to make the Reynolds number low. Then you can deduce the form of f using proportional reasoning.

You might think about these proportionalities:

1. How should Q depend on Δp ? For example, if you double the pressure difference, what should happen to the flow rate?
2. How should Q depend on l ? For example, if you keep the pressure difference the same but double the tube length, what happens to Q ? Or if you double Δp and double l , what happens to Q ?

Figure out the form of f to satisfy all your proportionality requirements.

If you get stuck going forward, instead work backward from the correct result. Look up Poiseuille flow, and use this result to deduce the preceding proportionalities; and then give reasons for why they are that way.

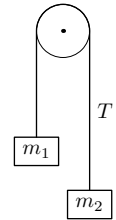
c. [optional]

The dimensional analysis in the preceding parts does not tell you the dimensionless constant. Use a syringe and needle to estimate the constant. Compare your constant with the value of $\pi/8$ that comes from solving the equations of fluid mechanics honestly.

Problem 7.8 Atwood machine: Tension in the string

Here is the Atwood machine from lecture. The string and pulley are massless and frictionless. We used dimensional analysis and special cases to guess the acceleration of either mass. With the right choice of sign,

$$\frac{a}{g} = \frac{m_1 - m_2}{m_1 + m_2}.$$



In this problem you guess the tension in the string.

- a. The tension T , like the acceleration, depends on m_1 , m_2 , and g . Explain why these four variables result in two independent dimensionless groups.
- b. Choose two suitable independent dimensionless groups so that you can write an equation for the tension in this form:

dimensionless group containing $T = f(\text{dimensionless group without } T)$.

The next part will be easier if you use a lot of symmetry in choosing the groups.

- c. Use special cases to guess f , and sketch f .
- d. Solve for T using the usual methods from introductory physics (8.01); then compare that answer with your answer from the preceding part.

Problem 7.9 Plant-watering system

The semester is over and you are going on holiday for a few weeks. But how will you water the house plants?! Design an unpowered slow-flow system to keep your plants happy.

Problem 7.10 Dimensional analysis for circuits

- a. Using Q as the dimension of charge, what are the dimensions of inductance L , capacitance C , and resistance R ?
 - b. Show that the dimensions of L , C , and R contain two independent dimensions.
 - c. In a circuit with one inductor, one capacitor, and one resistor, one dimensionless group should result from the three component values L , R , and C . What is physical interpretation of this group?
-

Chapter 8

Discretization

8.1 Random walks

Random walks are everywhere. Do you remember the card game War? How long does it last, on average? A molecule of neurotransmitter is released from a vesicle. Eventually it binds to the synapse, and your leg twitches. How long does it take to get there? On a winter day, you stand outside wearing only a thin layer of clothing. Why do you feel cold?

These physical situations are examples of random walks. In a physical random walk, for example a gas molecule moving and colliding, the walker moves a variable distance and can move in any direction. This general situation is complicated. Fortunately, the essential features of the random walk do not depend on these complicated details.

Simplify by discarding the generality. The generality arises from the continuous degrees of freedom: the direction is continuous and the distance between collisions is continuous. So, discretize the direction and the distance: Assume that the particle travels a fixed distance between collisions and that it can move only along the coordinate axes. Furthermore, analyze the special case of one-dimensional motion before going to the more general cases of two- and three-dimensional motion.

In this discretized, one-dimensional model, a particle starts at the origin and moves along a line. At each tick it moves left or right with probability $1/2$ in each direction. Let the position after n steps be x_n , and the expected position after n steps be $\langle x_n \rangle$. Because the random walk is unbiased – because moving in each direction is equally likely – the expected position remains constant:

$$\langle x_n \rangle = \langle x_{n-1} \rangle.$$

So $\langle x \rangle$, the so-called first moment of the position, is an invariant. However, it is not a fascinating invariant because it does not tell us much that we do not already understand intuitively.

Given that the first moment is not interesting, try the next-most-complicated moment: the second moment $\langle x^2 \rangle$. This analysis is easiest in special cases. Suppose that after a while wandering, the particle has arrived at 7, i.e. $x = 7$. At the next tick it will be at either $x = 6$ or $x = 8$. Its expected squared position – not its squared expected position! – has become

$$\langle x^2 \rangle = \frac{1}{2} (6^2 + 8^2) = 50.$$

The expected squared position increased by 1.

Let's check this pattern in a second example. Suppose that the particle is at $x = 10$, so $\langle x^2 \rangle = 100$. After one tick, the new expected squared position is

$$\langle x^2 \rangle = \frac{1}{2} (9^2 + 11^2) = 101.$$

Yet again $\langle x^2 \rangle$ has increased by 1! Based on those two examples, the conclusion is that

$$\langle x_{n+1}^2 \rangle = \langle x_n^2 \rangle + 1.$$

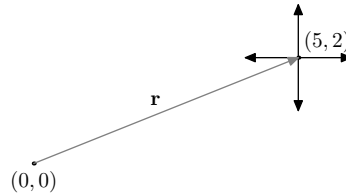
In other words,

$$\langle x_n^2 \rangle = n.$$

Since each step takes a constant time, in this discretized analysis, the conclusion is that

$$\langle x_n^2 \rangle \propto t.$$

The result that $\langle x^2 \rangle$ is proportional to time applied to the one-dimensional random walk. And it works for any dimension. Here's an example in two dimensions. Suppose that the particle's position is $(5, 2)$, so $\langle x^2 \rangle = 29$. After one step, it has four equally likely positions:



Rather than compute the new expected squared distance using all four positions, be lazy and just look at the two horizontal motions. The two possibilities are (6, 2) and (4, 2). The expected squared distance is

$$\langle x^2 \rangle = \frac{1}{2}(40 + 20) = 30,$$

which is one more than the previous value of $\langle x^2 \rangle$. Since nothing is special about horizontal motion compared to vertical motion – symmetry! – the same result holds for vertical motion. So, averaging over the four possible locations produces an expected squared distance of 30.

For two dimensions, the pattern is:

$$\langle x_{n+1}^2 \rangle = \langle x_n^2 \rangle + 1.$$

No step in the analysis depended on being in only two dimensions. In fancy words, the derivation and the result are invariant to change of dimensionality. In plain English, this result also works in three dimensions.

8.1.1 Difference between a random walk and a regular walk

In a standard walk in a straight line, $\langle x \rangle \propto \text{time}$. Note the single power of x . The interesting quantity in a regular walk is not x itself, since it can grow without limit and is not invariant, but the ratio x/t , which is invariant to changes in t . This invariant is also known as the speed.

In a random walk, where $\langle x^2 \rangle \propto t$, the interesting quantity is $\langle x^2 \rangle/t$. The expected squared position is not invariant to changes in t , but the ratio $\langle x^2 \rangle/t$ is invariant. This invariant is, except for a dimensionless constant, the *diffusion constant* often denoted D . It has dimensions of $L^2 T^{-1}$.

The difference between a random and a regular walk makes intuitive sense. A random walker, for example a gas molecule or a very drunk person, moves back and forth, sometimes making progress in one direction, and other times undoing that progress. So a random walker should take longer than a regular walker would take to travel the same distance. The relation $\langle x^2 \rangle/t \sim D$ confirms and sharpens this intuition. The time for a random walker to travel a distance l is $t \sim l^2/D$, which grows quadratically rather than linearly with distance.

8.1.2 Diffusion equation

The discretized model of a random explains where the diffusion equation comes from. Imagine a gas of particles with each particle doing a random

walk in one dimension. How does the concentration, or number, change with time?

Slice the one-dimensional world into slices of width Δx , and look at the slices at $x - \Delta x$, x , and $x + \Delta x$. In every time step, one-half the molecules in each slice move left, and one-half move right. So the number at x changes from $N(x)$ to

$$\frac{1}{2}(N(x - \Delta x) + N(x + \Delta x)),$$

for a change of

$$\begin{aligned}\Delta N &= \frac{1}{2}(N(x - \Delta x) + N(x + \Delta x)) - N(x) \\ &= \frac{1}{2}(N(x - \Delta x) - 2N(x) + N(x + \Delta x)).\end{aligned}$$

This last relation can be rewritten as

$$\Delta N \sim (N(x + \Delta x) - N(x)) - (N(x) - N(x + \Delta x)),$$

which in terms of derivatives is

$$\Delta N \sim (\Delta x)^2 \frac{\partial^2 N}{\partial x^2}.$$

The slices are separated by a distance such that most of the molecules travel from one piece to the neighboring piece in the time step τ . If τ is the time between collisions – the mean free time – then the distance is the mean free path λ . Thus

$$\frac{\Delta N}{\tau} \sim \frac{\lambda^2}{\tau} \frac{\partial^2 N}{\partial x^2},$$

or

$$\dot{N} \sim D \frac{\partial^2 N}{\partial x^2}$$

where $D \sim \lambda^2/\tau$ is a diffusion constant.

This partial-differential equation has interesting properties. The second spatial derivative means that a linear spatial concentration gradient remains unchanged: Its second derivative is zero so its time derivative must be zero. Diffusion smashes only curvature – roughly speaking, the second derivative – and does not try to change just the gradient. Heat often diffuses by a random walk, either via phonons (in a liquid or solid) or via molecular random walks (in a gas), so if you maintain one end of a bar at

T_1 and the other end at T_2 , then the bar will eventually linearly interpolate between the two temperatures, as long as heat is fed into the hot end and drawn out of the cold end.

8.1.3 Keeping warm

One consequence of random walks is how to keep warm on a cold day. We need to calculate the flux of heat: the energy flowing per unit area per unit time. We start from the definition of flux and reason physically.

Flux of stuff is defined as

$$\text{flux of stuff} = \frac{\text{stuff}}{\text{area} \times \text{time}}.$$

The flux depends on the density of stuff and on how fast the stuff travels:

$$\text{flux of stuff} = \frac{\text{stuff}}{\text{volume}} \times \text{speed}.$$

You can check that the dimensions are the same on both sides.

For heat flux, the stuff is thermal energy. The specific heat c_p is the thermal energy per mass, and $\rho c_p T$ is the thermal energy per volume. The speed is the 'speed' of diffusion. To diffuse a distance l takes time $t \sim l^2/D$, making the speed l/t or D/l . The l in the denominator indicates that, as expected, diffusion is slow over long distances. For heat diffusion, the diffusion constant is denoted κ and called the thermal diffusivity. So the speed is l/κ .

Combine the thermal energy per volume with the diffusion speed:

$$\text{thermal flux} = \rho c_p T \times \frac{\kappa}{l}.$$

The product $\rho c_p \kappa$ occurs so frequently that it is given a name: the thermal conductivity K . And the ratio T/l is a discretized version of the temperature gradient $\Delta T/\Delta x$. With those substitutions, the thermal flux is

$$F = K \frac{\Delta T}{\Delta x}.$$

To estimate how much heat one loses on a cold day, we need to estimate $K = \rho c_p \kappa$. Time to put all the pieces together for air:

$$\begin{aligned} \rho &\sim 1 \text{ kg m}^{-3}, \\ c_p &\sim 10^3 \text{ J kg}^{-1} \text{ K}^{-1}, \\ \kappa &\sim 1.5 \cdot 10^{-5} \text{ m}^2 \text{ s}^{-1}, \end{aligned}$$

where we are guessing that $\kappa = \nu$, since both are diffusion constants. Then

$$K = \rho c_p \kappa \sim 0.02 \text{ W m}^{-1} \text{ K}^{-1}.$$

Now we can estimate the heat loss outside on a cold day. Let's say that your skin is at 30°C and the air outside is 0°C , so $\Delta T = 30 \text{ K}$. A thin T-shirt may have thickness 2 mm , so

$$F = K \frac{\Delta T}{\Delta x} \sim 0.02 \text{ W m}^{-1} \text{ K}^{-1} \times \frac{30 \text{ K}}{2 \cdot 10^{-3} \text{ m}} \sim 300 \text{ W m}^{-2}.$$

Damn, I wanted a power not a power per area. Oh, flux is power per area, so all is well. I just need to multiply by my surface area. I'm roughly 2 m tall (approximately!) and 0.5 m wide, so my front and back each have area 1 m^2 . Then

$$P \sim FA = 300 \text{ W m}^{-2} \times 2 \text{ m}^2 = 600 \text{ W}.$$

No wonder it feels so cold! Just sitting around, your body generates 100 W (the basal metabolic rate). So, with 600 W escaping, you lose far more heat more than you generate. After long enough, your core body temperature drops. Chemical reactions in your body slow down, because all reactions go slower at lower temperature, and because enzymes lose their optimized shape. Eventually you die.

One solution is jogging to generate extra heat. That solution indicates that the estimate of 600 W is plausible. Cycling hard, which generates hundreds of watts of waste heat, is vigorous enough exercise to keep you warm, even on a winter day in thin clothing.

Another simple solution, as parents repeat to their children: Dress warmly by putting on thick layers. Let's recalculate the power loss if you put on a fleece that is 2 cm thick. You could redo the whole calculation from scratch, but it is simpler to notice that the thickness has gone up by a factor of 10. Since $F \propto 1/\Delta x$, the flux and the power drop by a factor of 10. So, when wearing the fleece,

$$P \sim 60 \text{ W}.$$

That heat loss is smaller than the basal metabolic rate, which indicates that you do not feel too cold. Indeed, when wearing a thick fleece, you feel most cold in your hands and face. Those regions are exposed to the air, and are protected by only a thin layer of still air. Because a small Δx means a large heat flux, the moral is: Listen to your parents, bundle up!

Problem 8.1 Perfume

If the diffusion constant (in air) for small perfume molecules is $10^{-6} \text{ m}^2 \text{ s}^{-1}$, estimate the time for perfume molecules to diffuse across a room.

Now try the experiment: How long does it take to smell the perfume from across the room? Explain the large discrepancy between the theoretical estimate and the experimental value.

Problem 8.2 Bending of light

In lecture we estimated how much gravity bends light by using dimensional analysis and then guessing the final functional form. In this problem, you analyze the same situation using discretization.

As before, let r be the closest that the light gets to the origin (which is the center of the gravitating mass). Discretize: Pretend that gravity forgets to deflect the photon unless the photon's distance to the origin is comparable to r . Using that idea, estimate the radial (inward) velocity imparted to the photon, and then the bending angle.

Feel free to neglect dimensionless constants like 2 or π , and check your answer against what we derived in lecture.

Problem 8.3 Teacup spindown

You stir your afternoon tea to mix the milk (and sugar if you have a sweet tooth). Once you remove the stirring spoon, the rotation starts to slow. What is the spindown time τ ? In other words, how long before the angular velocity of the tea has fallen by a significant fraction?

To estimate τ , consider a physicist's idea of a teacup: a cylinder with height L and diameter L , filled with liquid. Why does the rotation slow? Tea near the edge of the teacup – and near the base, but for simplicity neglect the effect of the base – is slowed by the presence of the edge (the no-slip boundary condition). The edge produces a velocity gradient.

Because of the tea's viscosity, the velocity gradient produces a force on any piece of the edge. This force tries to spin the piece in the direction of the tea's motion. The piece exerts a force on the tea equal in magnitude and opposite in direction. Therefore, the edge slows the rotation. Now you can analyze this model quantitatively.

- In terms of the total viscous force F and of the initial angular velocity ω , estimate the spindown time. Hint: Consider torque and angular momentum. (Feel free to drop any constants, such as π and 2, by invoking the Estimation Theorem: $1 = 2$.)
- You can estimate F with the idea that

$$\text{viscous force} \sim \rho v \times \text{velocity gradient} \times \text{surface area}.$$

Here $\rho\nu$ is η . The more familiar viscosity is η , known as the dynamic viscosity. The more convenient viscosity is ν , the kinematic viscosity. The velocity gradient is determined by the size of the region in which the the edge has a significant effect on the flow; this region is called the boundary layer. Let δ be its thickness. Estimate the velocity gradient near the edge in terms of δ , and use the equation for viscous force to estimate F .

- c. Put your expression for F into your earlier estimate for τ , which should now contain only one quantity that you have not yet estimated (the boundary-layer thickness).
- d. You can estimate δ using your knowledge of random walks. The boundary layer is a result of momentum diffusion; just as D is the molecular-diffusion coefficient, ν is the momentum-diffusion coefficient. In a time t , how far can momentum diffuse? This distance is δ . What is a natural estimate for t ? (Hint: After rotating 1 radian, the fluid is moving in a significantly different direction than before, so the momentum fluxes no longer add.) Use that time to estimate δ .
- e. Now put it all together: What is the characteristic spindown time τ (the time for the rotation to slow down by a significant amount)?
- f. Stir some tea to experimentally estimate τ_{exp} . Compare this time with the time predicted by the preceding theory. [In water (and tea is roughly water), $\nu \sim 210^{-6} \text{ m}^2 \text{ s}^{-1}$.]

Problem 8.4 Stokes' law

You can use ideas from **Problem 8.3** to derive Stokes' formula for drag at low speeds (more precisely, at low Reynolds' number). In the text we derived the result from dimensional analysis; here you will develop a physical argument.

Consider a sphere of radius R moving with velocity v . Equivalently, in the reference frame of the sphere, the sphere is fixed and the fluid moves past it with velocity v . Next to the sphere, the fluid is stationary. Over a region of thickness δ (the boundary layer), the fluid velocity rises from zero to the full flow speed v . Assume that $\delta \sim R$ (the most natural assumption) and estimate the viscous drag force. Compare the force with Stokes' formula (remember that $\rho\nu = \eta$).

Problem 8.5 Bouncing ball

You drop a steel ball, say $r \sim 1 \text{ cm}$, from a height of one or two metres. It lands on a steel surface and bounces up to nearly the original drop height. Estimate the contact force at the instant when the ball is stationary on the ground. Give your answer as the dimensionless ratio

$$\frac{\text{contact force}}{\text{weight of the ball}}.$$

Useful data: The elastic modulus of steel is $Y \sim 10^{11} \text{ N m}^{-2}$.

Problem 8.6 Cone free-fall time and distance

Estimate how long the falling cones of [Section 4.3.1](#) require to reach (a good fraction of) terminal velocity. And estimate how far they fall before reaching (a good fraction of) terminal velocity.

Chapter 9

Successive approximation

Chapter 10

Springs

Every physical process contains a spring! The main example in this chapter is waves, which illustrate springs, discretization, and special cases – a fitting, unified way to end the book.

10.1 Musical tones

10.1.1 Wood blocks

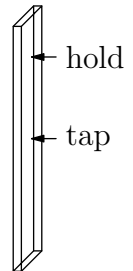
Here is a home musical experiment that illustrates proportional reasoning and springs. First construct, or ask a carpenter or a local lumber yard to construct, two wood blocks made from the same larger wood plank. Mine have these dimensions:

1. 30 cm $\times$ 5 cm $\times$ 1 cm; and
2. 30 cm $\times$ 5 cm $\times$ 2 cm.

The blocks are identical except in their thickness: 2 cm vs 1 cm.

Now tap the thin block at the center while holding it lightly toward the edge, and listen to the musical note. If you do the same experiment to the thick block, will the pitch (fundamental frequency) be higher than, the same as, or lower than the pitch when you tapped the thin block?

You can answer this question in many ways. The first is to do the experiment. It would be nice either to predict the result before doing the experiment or to explain and understand the result after doing the experiment.

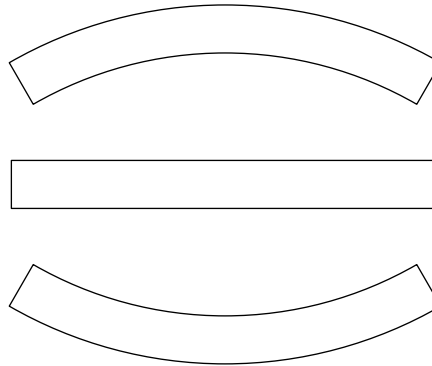


One argument is that the block is a resonant object, and the wavelength of the sound depends on the thickness of the block. In that picture, the thick

block should have the longer wavelength and therefore the lower frequency. A counterargument, based on a different model of how the sound is made, is that the thick block is stiffer, so it vibrates faster. On the other hand, the thick block is more massive, so it vibrates more slowly. Perhaps these two effects – greater stiffness but greater mass – cancel each other, leaving the frequency unchanged?

I'll do the experiment right now and tell you the result. The thick block has a higher pitch. So the resonant-cavity model is probably wrong. Instead, the stiffness probably more than overcomes the mass.

A spring model explains this result and even predicts the frequency ratio. In the spring model, a wood block is made of wood atoms connected by chemical bonds, which are springs. As the block vibrates, it takes shapes like these (in a side view):



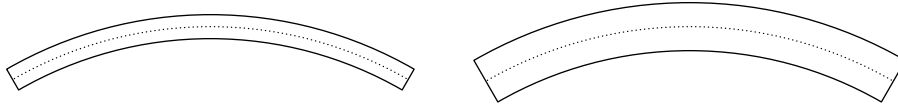
The block is made of springs, and it acts like a big spring. The middle position is the equilibrium position, when the block has zero potential energy and maximum kinetic energy. The potential energy is stored in stretching and compressing the bonds. Imagine deforming the block into a shape like the top shape. Since the block is a big spring, to produce the vertical deflection y requires an energy $E \sim ky^2$, where k is the stiffness of the block.

To find how k depends on the thickness h , deflect the thin and thick blocks by the same amount y , and compare the stored energies:

$$\frac{k_{\text{thick}}}{k_{\text{thin}}} = \frac{E_{\text{thick}}}{E_{\text{thin}}},$$

because y is, by construction, the same for the thick and thin blocks.

Here are the blocks, with the dotted line showing the neutral line, which is the line without compression or extension:



Above the neutral line the springs are extended. Below the neutral line, the springs are compressed. The amount of extension is proportional to the distance from the neutral line. Each spring in the thin block corresponds to a spring in the thick block that is twice as far away from the neutral line. The spring in the thick block has twice the extension (or compression) of its partner in the thin block. So the spring in the thick block stores four times the energy of its partner spring in the thin block. Furthermore, the thick block has twice as many layers as does the thin block, so each spring in the thin block has two partners, with identical extension, in the thick block. So the thick block stores eight times the energy of the thin block, for the same deflection y .

Thus

$$\frac{k_{\text{thick}}}{k_{\text{thin}}} = 8.$$

This factor of 8 results from multiplying the thickness by 2. In general,

$$k \propto h^3.$$

Since $\omega \sim \sqrt{k/m}$, and

$$\frac{m_{\text{thick}}}{m_{\text{thin}}} = 2,$$

the frequency ratio is

$$\frac{\omega_{\text{thick}}}{\omega_{\text{thin}}} = \sqrt{\frac{8}{2}} = 2.$$

In general, $m \propto h$ so

$$\frac{\omega_{\text{thick}}}{\omega_{\text{thin}}} = \sqrt{\frac{h^3}{h}} = h.$$

Frequency is proportional to thickness!

Let's check this analysis by looking at its consequences and comparing with experimental data from a home experiment.

10.1.2 Xylophone

My daughter got a toy xylophone from her uncle. Its slats have these dimensions:

	ℓ
C	12.2 cm
D	11.5
E	10.9
F	10.6
G	10.0
A	9.4
B	8.9
C'	8.6

Our analysis of how frequency depends on thickness can explain this pattern of how frequency depends on length. The method is to use dimensional analysis with proportional reasoning (scaling).

Rather than finding the frequency direction, I analyze the stiffness. The mass is easy, so split that part off of the calculation of the frequency. The block's spring constant k depends on its material properties – here, the Young's modulus Y – and on its dimensions. So the variables are k , Y , and length l , width w , and thickness (height) h .

- *How many independent dimensions are contained in those variables? How many independent dimensionless groups can be formed from those variables?*

These five variables are composed of two independent dimensions. These dimensions could be length and force: Stiffness is force per length, and Young's modulus is force per area. Five variables based on two independent dimensions form three independent dimensionless groups. The goal is to find k , so I include k in only one group. That group contains Y to divide out the dimensions of mass. Since Young's modulus is force per area, and stiffness is force per length, the ratio k/Yh is dimensionless. The three lengths for the size of the block easily make two more dimensionless groups: for example, h/l and w/l . Then

$$\frac{k}{Yh} = f\left(\frac{h}{l}, \frac{w}{l}\right).$$

- *Guess the function f (except for a dimensionless constant).*

What we know about stiffness versus thickness, along with proportional reasoning, is enough to solve for f , except for a dimensionless constant.

Proportional reasoning helps determine the dependence on the dimensionless group w/l . Imagine doubling the width w . Equivalently, I glue together two identical blocks along the long, thin edge. When the new block is bent, the individual blocks contains equal energy, so the new block contains twice the energy of an original block. Therefore, doubling the width doubles the stiffness; in symbols, $k \propto w$. In the general form

$$\frac{k}{Yh} = f\left(\frac{h}{l}, \frac{w}{l}\right).$$

w appears only in the group w/l , and k appears on the right in the first power. So the general form simplifies to

$$\frac{k}{Yh} = \frac{w}{l} \cdot f\left(\frac{h}{l}\right).$$

To guess the new function f , I use what I know about stiffness versus thickness, that $k \propto h^3$. Therefore the left side, k/Yh , is proportional to h^2 . On the right side the only source of h is from f , which can play with h but only via the ratio h/l . So

$$f\left(\frac{h}{l}\right) \sim \left(\frac{h}{l}\right)^2.$$

Combining these deductions gives

$$\frac{k}{Yh} \sim \frac{w}{l} \left(\frac{h}{l}\right)^2 = \frac{wh^2}{l^3}$$

and

$$k \sim Yw \left(\frac{h}{l}\right)^3.$$

The stiffness and mass determine the frequency. The mass is $m = \rho wlh$. So

$$\omega \sim \sqrt{\frac{k}{m}} \sim \sqrt{\frac{Y}{\rho}} \frac{h}{l^2}.$$

As a quick check, this result is consistent with the earlier calculation that frequency is proportional to thickness. And it contains a new result: $\omega \propto l^{-2}$.

Problem 10.1 Effect of width

Is it physically plausible that the width w does not affect the frequency ω ?

► *Is this data consistent with the prediction that $\omega \propto l^{-2}$?*

Before doing an extensive analysis, I check the easy case of the octave. The lower and higher C notes are a factor of 2 apart in frequency. If the scaling prediction is correct, the respective slat lengths should be a factor of $\sqrt{2}$ apart. The length ratio is $12.2/8.6 \sim 1.419$, which is very close to $\sqrt{2}$. The general pattern is that fl^2 should be invariant. To check, here is the same table with frequencies, which are computed by assuming that the A above middle C is at 440 Hz (concert A), and with a column for fl^2 :

	l	f
C	12.2	261.6
D	11.5	293.6
E	10.9	329.6
F	10.6	349.2
G	10.0	392.0
A	9.4	440.0
B	8.9	493.8
C'	8.6	523.2

The proposed invariant is, experimentally, almost constant.

10.2 Waves

Ocean covers most of the earth, and waves roam most of the ocean. Waves also cross puddles and ponds. What makes them move, and what determines their speed? By applying and extending the techniques of approximation, we analyze waves. For concreteness, this section refers mostly to water waves but the results apply to any fluid. The themes of section are: *Springs are everywhere* and *Consider limiting cases*.

10.2.1 Dispersion relations

The most organized way to study waves is to use **dispersion relations**. A dispersion relation states what values of frequency and wavelength a wave can have. Instead of the wavelength λ , dispersion relations usually connect frequency ω , and wavenumber k , which is defined as $2\pi/\lambda$. This preference has an basis in order-of-magnitude reasoning. Wavelength is the the distance the wave travels in a full period, which is 2π radians of oscillation. Although 2π is dimensionless, it is not the ideal dimensionless number, which is unity. In 1 radian of oscillation, the wave travels a distance

$$\bar{\lambda} \equiv \frac{\lambda}{2\pi}.$$

The bar notation, meaning ‘divide by 2π ’, is chosen by analogy with $\hbar$ and $\hbar$. The one-radian forms such as $\hbar$ are more useful for approximations than the 2π -radian forms. The Bohr radius, in a form where the dimensionless constant is unity, contains $\hbar$ rather than h . Most results with waves are similarly simpler using $\bar{\lambda}$ rather than λ . A further refinement is to use its inverse, the wavenumber $k = 1/\bar{\lambda}$. This choice, which has dimensions of inverse length, parallels the definition of angular frequency ω , which has dimensions of inverse time. A relation that connects ω and k is likely to be simpler than one connecting ω and $\bar{\lambda}$, although either is simpler than one connecting ω and λ .

The simplest dispersion relation describes electromagnetic waves in a vacuum. Their frequency and wavenumber are related by the dispersion relation

$$\omega = ck,$$

which states that waves travel at velocity $\omega/k = c$, independent of frequency. Dispersion relations contain a vast amount of information about waves. They contain, for example, how fast crests and troughs travel: the **phase velocity**. They contain how fast wave packets travel: the **group velocity**. They contain how these velocities depend on frequency: the **dispersion**. And they contain the rate of energy loss: the **attenuation**.

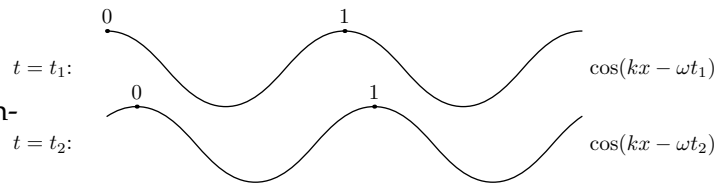
10.2.2 Phase and group velocities

The usual question with a wave is how fast it travels. This question has two answers, the phase velocity and the group velocity, and both depend on the dispersion relation. To get a feel for how to use dispersion relations (most of the chapter is about how to calculate them), we discuss the simplest examples that illustrate these two velocities. These analyses start with the general form of a traveling wave:

$$f(x, t) = \cos(kx - \omega t),$$

where f is its amplitude.

Phase velocity is an easier idea than group velocity so, as an example of divide-and-conquer reasoning and of maximal laziness, study it first. The phase, which is the argument of the cosine, is $kx - \omega t$. A crest occurs when the phase is zero. In the top wave, a crest occurs when $x = \omega t_1/k$. In the bottom wave, a crest occurs when $x = \omega t_2/k$. The difference



$$\frac{\omega}{k}(t_2 - t_1)$$

is the distance that the crest moved in time $t_2 - t_1$. So the phase velocity, which is the velocity of the crests, is

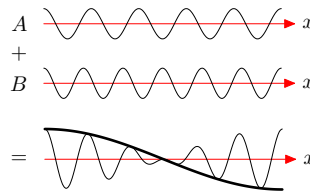
$$v_{ph} = \frac{\text{distance crest shifted}}{\text{time taken}} = \frac{\omega}{k}.$$

To check this result, check its dimensions: ω provides inverse time and $1/k$ provides length, so ω/k is a speed.

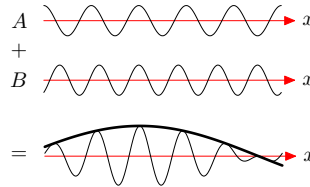
Group velocity is trickier. The word ‘group’ suggests that the concept involves more than one wave. Because two is the first whole number larger than one, the simplest illustration uses two waves. Instead of being a function relating ω and k , the dispersion relation here is a list of allowed (k, ω) pairs. But that form is just a discrete approximation to an official dispersion relation, complicated enough to illustrate group velocity and simple enough to not create a forest of mathematics. So here are two waves with almost the same wavenumber and frequency:

$$\begin{aligned} f_1 &= \cos(kx - \omega t), \\ f_2 &= \cos((k + \Delta k)x - (\omega + \Delta \omega)t), \end{aligned}$$

where Δk and $\Delta \omega$ are small changes in wavenumber and frequency, respectively. Each wave has phase velocity $v_{ph} = \omega/k$, as long as Δk and $\Delta \omega$ are tiny. The figure shows their sum.



The point of the figure is that two cosines with almost the same spatial frequency add to produce an envelope (thick line). The envelope itself looks like a cosine. After waiting a while, each wave changes because of the ωt or $(\omega + \Delta\omega)t$ terms in their phases. So the sum and its envelope change to this:



The envelope moves, like the crests and troughs of any wave. It is a wave, so it has a phase velocity, which motivates the following definition:

Group velocity is the phase velocity of the envelope.

With this pictorial definition, you can compute group velocity for the wave $f_1 + f_2$. As suggested in the figures, adding two cosines produces a slowly varying envelope times a rapidly oscillating inner function. This trigonometric identity gives the details:

$$\cos(A + B) = \underbrace{2 \cos\left(\frac{B - A}{2}\right)}_{\text{envelope}} \times \underbrace{\cos\left(\frac{A + B}{2}\right)}_{\text{inner}}.$$

If $A \approx B$, then $A - B \approx 0$, which makes the envelope vary slowly. Apply the identity to the sum:

$$f_1 + f_2 = \underbrace{\cos(kx - \omega t)}_A + \underbrace{\cos((k + \Delta k)x - (\omega + \Delta\omega)t)}_B.$$

Then the envelope contains

$$\cos\left(\frac{B - A}{2}\right) = \cos\left(\frac{x\Delta k - t\Delta\omega}{2}\right).$$

The envelope represents a wave with phase

$$\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t.$$

So it is a wave with wavenumber $\Delta k/2$ and frequency $\Delta\omega/2$. The envelope's phase velocity is the group velocity of $f_1 + f_2$:

$$v_g = \frac{\text{frequency}}{\text{wavenumber}} = \frac{\Delta\omega/2}{\Delta k/2} = \frac{\Delta\omega}{\Delta k}.$$

In the limit where $\Delta k \rightarrow 0$ and $\Delta\omega \rightarrow 0$, the group velocity is

$$v_g = \frac{\partial\omega}{\partial k}.$$

It is usually different from the phase velocity. A typical dispersion relation, which appears several times in this chapter, is $\omega \propto k^n$. Then $v_{ph} = \omega/k = k^{n-1}$ and $v_g \propto nk^{n-1}$. So their ratio is

$$\frac{v_g}{v_{ph}} = n. \quad (\text{for a power-law relation})$$

Only when $n = 1$ are the two velocities equal. Now that we can find wave velocities from dispersion relations, we return to the problem of finding the dispersion relations.

10.2.3 Dimensional analysis

A dispersion relation usually emerges from solving a wave equation, which is an unpleasant partial differential equation. For water waves, a wave equation emerges after linearizing the equations of hydrodynamics and neglecting viscosity. This procedure is mathematically involved, particularly in handling the boundary conditions. Alternatively, you can derive dispersion relations using dimensional analysis, then complete and complement the derivation with physical arguments. Such methods usually cannot evaluate the dimensionless constants, but the beauty of studying waves is that, as in most problems involving springs and oscillations, *most of these constants are unity*.

How do frequency and wavenumber connect? They have dimensions of T^{-1} and L^{-1} , respectively, and cannot form a dimensionless group without help. So include more variables. What physical properties of the system determine wave behavior? Waves on the open ocean behave differently from waves in a bathtub, perhaps because of the difference in the depth of water h . The width of the tub or ocean could matter, but then the problem becomes two-dimensional wave motion. In a first analysis, avoid that complication and consider waves that move in only one dimension, perpendicular to the width of the container. Then the width does not matter.

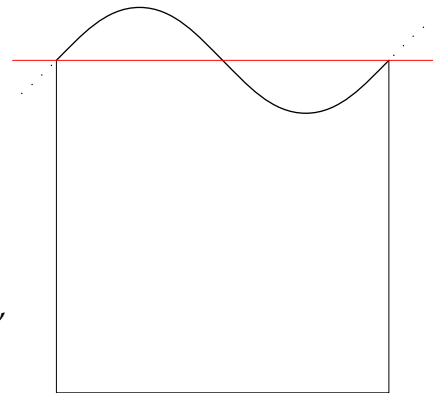
To determine what other variables are important, use the principle that waves are like springs, because *every physical process contains a spring*. This blanket statement cannot be strictly correct. However, it is useful as a broad

generalization. To get a more precise idea of when this assumption is useful, consider the characteristics of spring motion. First, springs have an equilibrium position. If a system has an undisturbed, resting state, consider looking for a spring. For example, for waves on the ocean, the undisturbed state is a calm, flat ocean. For electromagnetic waves – springs are not confined to mechanical systems – the resting state is an empty vacuum with no radiation. Second, springs oscillate. In mechanical systems, oscillation depends on inertia to carry the mass beyond the equilibrium position. Equivalently, it depends on kinetic energy turning into potential energy, and vice versa. Water waves store potential energy in the disturbance of the surface and kinetic energy in the motion of the water. Electromagnetic waves store energy in the electric and magnetic fields. A magnetic field is generated by moving or spinning charges, so the magnetic field is a reservoir of kinetic (motion) energy. An electric field is generated by stationary charges and has an associated potential, so the electric field is the reservoir of potential energy. With these identifications, the electromagnetic field acts like a set of springs, one for each radiation frequency. These examples are positive examples. A negative example – a marble oozing its way through glycerin – illustrates that springs are not always present. The marble moves so slowly that the kinetic energy of the corn syrup, and therefore its inertia, is miniscule and irrelevant. This system has no reservoir of kinetic energy, for the kinetic energy is merely dissipated, so it does not contain a spring.

Waves have the necessary reservoirs to act like springs. The surface of water is flat in its lowest-energy state. Deviations, also known as waves, are opposed by a restoring force. Distorting the surface is like stretching a rubber sheet: Surface tension resists the distortion. Distorting the surface also requires raising the average water level, a change that gravity resists.

The average *height* of the surface does not change, but the average depth of the water does. The higher column, under the crest, has more water than the lower column, under the trough. So in averaging to find the average depth, the higher column gets a slightly higher weighting. Thus the average depth increases. This result is consistent with experience: It takes energy to make waves.

The total restoring force includes gravity and surface tension so, in the list of variables, include surface tension (γ) and gravity (g).



In a wave, like in a spring, the restoring force fights inertia, represented here by the fluid density. The gravitational piece of the restoring force does not care about density: Gravity's stronger pull on denser substances is exactly balanced by their greater inertia. This exact cancellation is a restatement of the **equivalence principle**, on which Einstein based the theory of general relativity [8, 9]. In pendulum motion, the mass of the bob drops out of the final solution for the same reason. The surface-tension piece of the restoring force, however, does not change when density changes. The surface tension itself, the fluid property γ , depends on density because it depends on the spacing of atoms at the surface. That dependence affects γ . However, once you know γ you can compute surface-tension forces without knowing the density. Since ρ does not affect the surface-tension force but affects the inertia, it affects the properties of waves in which surface tension provides a restoring force. Therefore, include ρ in the list.

To simplify the analysis, assume that the waves do not lose energy. This choice excludes viscosity from the set of variables. To further simplify, exclude the speed of sound. This approximation means ignoring sound waves, and is valid as long as the flow speeds are slow compared to the speed of sound. The resulting ratio,

$$\mathcal{M} \equiv \frac{\text{flow speed}}{\text{sound speed}},$$

Var	Dim	What
ω	T^{-1}	frequency
k	L^{-1}	wavenumber
g	LT^{-2}	gravity
h	L	depth
ρ	ML^{-3}	density
γ	MT^{-2}	surface tension

is dimensionless and, because of its importance, is given a name: the **Mach number**. Finally, assume that the wave amplitude ξ is small compared to its wavelength and to the depth of the container. The table shows the list of variables. Even with all these restrictions, which significantly simplify the analysis, the results explain many phenomena in the world.

These six variables built from three fundamental dimensions produce three dimensionless groups. One group is easy: the wavenumber k is an inverse length and the depth h is a length, so

$$\Pi_1 \equiv kh.$$

This group is the dimensionless depth of the water: $\Pi_1 \ll 1$ means shallow and $\Pi_1 \gg 1$ means deep water. A second dimensionless group comes from gravity. Gravity, represented by g , has the same dimensions as ω^2 , except for a factor of length. Dividing by wavenumber fixes this deficit:

$$\Pi_2 = \frac{\omega^2}{gk}.$$

Without surface tension, Π_1 and Π_2 are the only dimensionless groups, and neither group contains density. This mathematical result has a physical basis. Without surface tension, the waves propagate because of gravity alone. The equivalence principle says that gravity affects motion independently of density. Therefore, density cannot – and does not – appear in either group.

Now let surface tension back into the playpen of dimensionless groups. It must belong in the third (and final) group Π_3 . Even knowing that γ belongs to Π_3 still leaves great freedom in choosing its form. The usual pattern is to find the group and then interpret it, as we did for Π_1 and Π_2 . Another option is to begin with a physical interpretation and use the interpretation to construct the group. Here you can construct Π_3 to measure the relative importance of surface-tension and gravitational forces. Surface tension γ has dimensions of force per length, so producing a force requires multiplying by a length. The problem already has two lengths: wavelength (represented via k) and depth. Which one should you use? The wavelength probably always affects surface-tension forces, because it determines the curvature of the surface. The depth, however, affects surface-tension forces only when it becomes comparable to or smaller than the wavelength, if even then. You can use both lengths to make γ into a force: for example, $F \sim \gamma \sqrt{h/k}$. But the analysis is easier if you use only one, in which case the wavelength is the preferable choice. So $F_\gamma \sim \gamma/k$. Gravitational force, also known as weight, is $\rho g \times \text{volume}$. Following the precedent of using only k to produce a length, the gravitational force is $F_g \sim \rho g/k^3$. The dimensionless group is then the ratio of surface-tension to gravitational forces:

$$\Pi_3 \equiv \frac{F_\gamma}{F_g} = \frac{\gamma/k}{\rho g/k^3} = \frac{\gamma k^2}{\rho g}.$$

This choice has, by construction, a useful physical interpretation, but many other choices are possible. You can build a third group without using gravity: for example, $\Pi_3 \equiv \gamma k^3/\rho \omega^2$. With this choice, ω appears in two groups: Π_2 and Π_3 . So it will be hard to solve for it. The choice made for Π_3 , besides being physically useful, quarantines ω in one group: a useful choice since ω is the goal.

Now use the groups to solve for frequency ω as a function of wavenumber k . You can instead solve for k as a function of ω , but the formulas for phase and group velocity are simpler with ω as a function of k . Only the group Π_2 contains ω , so the general dimensionless solution is

$$\Pi_2 = f(\Pi_1, \Pi_3),$$

or

$$\frac{\omega^2}{gk} = f\left(kh, \frac{\gamma k^2}{\rho g}\right).$$

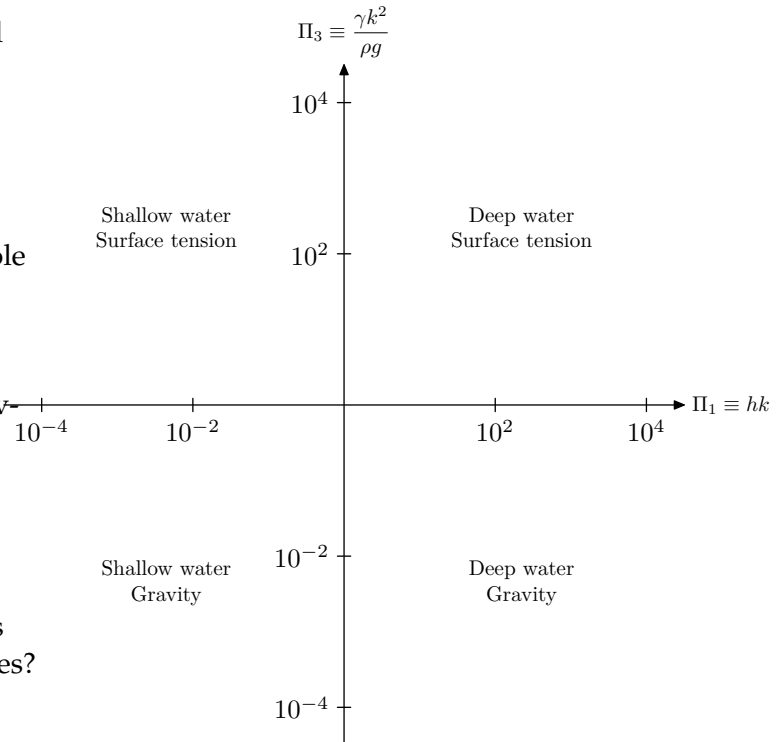
Then

$$\omega^2 = gk \cdot f\left(kh, \frac{\gamma k^2}{\rho g}\right).$$

This relation is valid for waves in shallow or deep water (small or large Π_1); for waves propagated by gravity or by surface tension (small or large Π_3); and for waves in between.

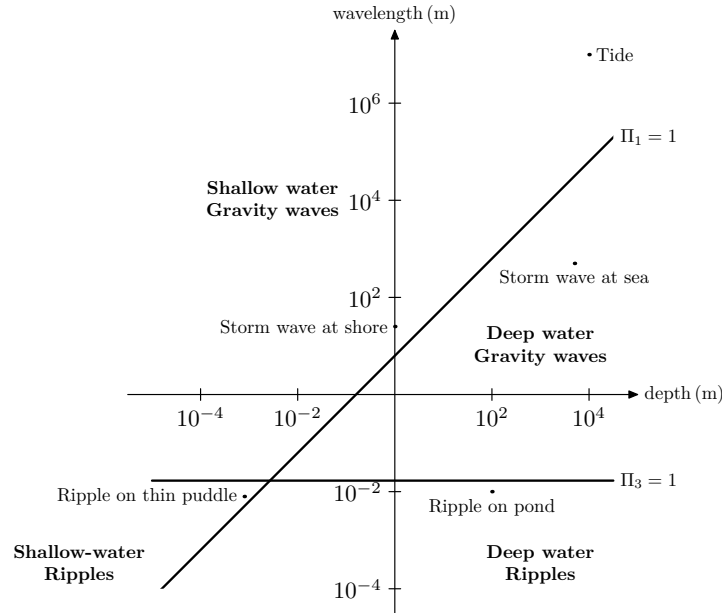
The figure shows how the two groups Π_1 and Π_3 divide the world of waves into four regions. We study each region in turn, and combine the analyses to understand the whole world (of waves). The group Π_1 measures the depth of the water: Are the waves moving on a puddle or an ocean? The group Π_3 measures the relative contribution of surface tension and gravity: Are the waves ripples or gravity waves?

The division into deep and shallow water (left and right sides) follows from the interpretation of $\Pi_1 = kh$ as dimensionless depth. The division into surface-tension- and gravity-dominated waves (top and bottom halves) is more subtle, but is a result of how Π_3 was constructed. As a check, look at Π_3 . Large g or small k result in the same consequence: small Π_3 . Therefore the physical consequence of longer



wavelength (smaller k) is similar to that of stronger gravity. So longer-wavelength waves are gravity waves. The large- Π_3 portion of the world (top half) is therefore labeled with surface tension.

The next figure shows how wavelength and depth (instead of the dimensionless groups) partition the world, and plots examples of different types of waves.

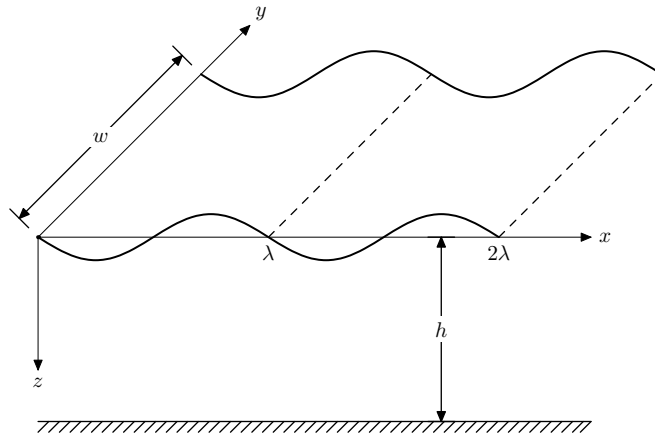
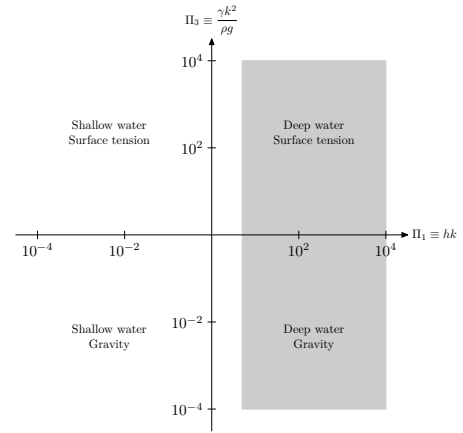


The thick dividing lines are based on the dimensionless groups $\Pi_1 = hk$ and $\Pi_3 = \gamma k^2 / \rho g$. Each region contains one or two examples of its kind of waves. With $g = 1000 \text{ cm s}^{-1}$ and $\rho \sim 1 \text{ g cm}^{-3}$, the border wavelength between ripples and gravity waves is just over $\lambda \sim 1 \text{ cm}$ (the horizontal, $\Pi_3 = 1$ dividing line).

The magic function f is still unknown to us. To determine its form and to understand its consequences, study f in limiting cases. Like a jigsaw-puzzle-solver, study first the corners of the world, where the physics is simplest. Then connect the corner solutions to get solutions valid along an edge, where the physics is the almost as simple as in a corner. Finally, crawl inward to assemble the complicated, complete solution. This extended example illustrates divide-and-conquer reasoning, and using limiting cases to choose pieces into which you break the problem.

10.2.4 Deep water

First study deep water, where $kh \gg 1$, as shaded in the map. Deep water is defined as water sufficiently deep that waves cannot feel the bottom of the ocean. How deep do waves' feelers extend? The only length scale in the waves is the wavelength, $\lambda = 2\pi/k$. The feelers therefore extend to a depth $d \sim 1/k$ (as always, neglect constants, such as 2π). This educated guess has a justification in Laplace's equation, which is the spatial part of the wave equation. Suppose that the waves are periodic in the x direction, and z measures depth below the surface, as shown in this figure:



Then, Laplace's equation becomes

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0,$$

where ϕ is the velocity potential. The $\partial^2 \phi / \partial y^2$ term vanishes because nothing varies along the width (the y direction).

It's not important what exactly ϕ is. You can find out more about it in an excellent fluid-mechanics textbook, *Fluid Dynamics for Physicists* [10]; Lamb's *Hydrodynamics* [19] is a classic but difficult. For this argument, all that matters is that ϕ measures the effect of the wave and that ϕ satisfies Laplace's equation. The wave is periodic in the x direction, with a form such as $\sin kx$. Take

$$\phi \sim Z(z) \sin kx.$$

The function $Z(z)$ measures how the wave decays with depth.

The second derivative in x brings out two factors of k , and a minus sign:

$$\frac{\partial^2 \phi}{\partial x^2} = -k^2 \phi.$$

In order that this ϕ satisfy Laplace's equation, the z -derivative term must contribute $+k^2 \phi$. Therefore,

$$\frac{\partial^2 \phi}{\partial z^2} = k^2 \phi,$$

so $Z(z) \sim e^{\pm kz}$. The physically possible solution – the one that does not blow up exponentially at the bottom of the ocean – is $Z(z) \sim e^{-kz}$. Therefore, relative to the effect of the wave at the surface, the effect of the wave at the bottom of the ocean is $\sim e^{-kh}$. When $kh \gg 1$, the bottom might as well be on the moon because it has no effect. The dimensionless factor kh – it must be dimensionless to sit alone in an exponent – compares water depth with feeler depth $d \sim 1/k$:

$$\frac{\text{water depth}}{\text{feeler depth}} \sim \frac{h}{1/k} = hk,$$

which is the dimensionless group Π_1 .

In deep water, where the bottom is hidden from the waves, the water depth h does not affect their propagation, so h disappears from the list of relevant variables. When h goes, so does $\Pi_1 = kh$. There is one caveat. If Π_1 is the only group that contains k , then you cannot blithely discard Π_1 just because you no longer care about h . If you did, you would be discarding k and h , and make it impossible to find a dispersion relation (which connects ω and k). Fortunately, k appears in $\Pi_3 = \gamma k^2 / \rho g$ as well as in Π_1 . So in deep water it is safe to discard Π_1 . This argument for the irrelevance of h is a physical argument. It has a mathematical equivalent in the language of dimensionless groups and functions. Because h has dimensions, the statement that ' h is large' is meaningless. The question is, 'large compared to what length?' With $1/k$ as the standard of comparison the meaningless ' h is large' statement becomes ' kh is large.' The product kh is the dimensionless group Π_1 . Mathematically, you are assuming that the function $f(kh, \gamma k^2 / \rho g)$ has a limit as $kh \rightarrow \infty$.

Without Π_1 , the general dispersion relation simplifies to

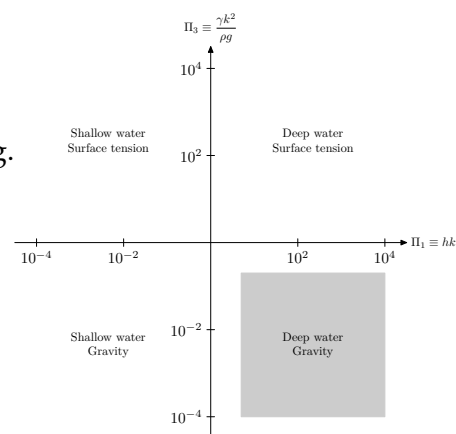
$$\omega^2 = gk f_{\text{deep}} \left(\frac{\gamma k^2}{\rho g} \right).$$

This relation describes the deep-water edge of the world of waves. The edge has two corners, labeled by whether gravity or surface tension provides the restoring force. Although the form of f_{deep} is unknown, it is a simpler function than the original f , a function of two variables. To determine the form of f_{deep} , continue the process of dividing and conquering: Partition deep-water waves into its two limiting cases, gravity waves and ripples.

10.2.5 Gravity waves on deep water

Of the two extremes, gravity waves are the more common. They include wakes generated by ships and most waves generated by wind. So specialize to the corner of the wave world where water is deep and gravity is strong. With gravity much stronger than surface tension, the dimensionless group $\Pi_3 = \gamma k^2 / \rho g$ limits to 0. Since Π_3 is the product of several factors, you can achieve the limit in several ways:

1. Increase g (which is downstairs) by moving to Jupiter.
2. Reduce γ (which is upstairs) by dumping soap on the water.
3. Reduce k (which is upstairs) by studying waves with a huge wavelength.



In this limit, the general deep-water dispersion relation simplifies to

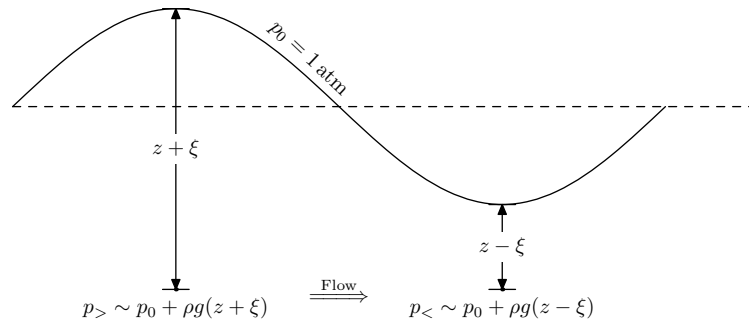
$$\omega^2 = f_{\text{deep}}(0) gk = C_1 gk,$$

where $f_{\text{deep}}(0)$ is an as-yet-unknown constant, C_1 . The use of $f_{\text{deep}}(0)$ assumes that $f_{\text{deep}}(x)$ has a limit as $x \rightarrow 0$. The slab argument, which follows shortly, shows that it does. For now, in order to make progress, assume that it has a limit. The constant remains unknown to the lazy methods of dimensional analysis, because the methods sacrifice evaluation of dimensionless constants to gain comprehension of physics. Usually assume that such constants are unity. In this case, we get lucky: An honest calculation produces $C_1 = 1$ and

$$\omega^2 = 1 \times gk,$$

where the red $1 \times$ indicates that it is obtained from honest physics.

Such results from dimensional analysis seem like rabbits jumping from a hat. The dispersion relation is correct, but your gut may grumble about this magical derivation and ask, ‘But *why* is the result true?’ A physical model of the forces or energies that drive the waves explains the origin of the dispersion relation. The first step is to understand the mechanism: How does gravity make the water level rise and fall? Taking a hint from the Watergate investigators,¹ we follow the water. The water in the crest does *not* move into the trough. Rather, the water in the crest, being higher, creates a pressure underneath it higher than that of the water in the trough, as shown in this figure:

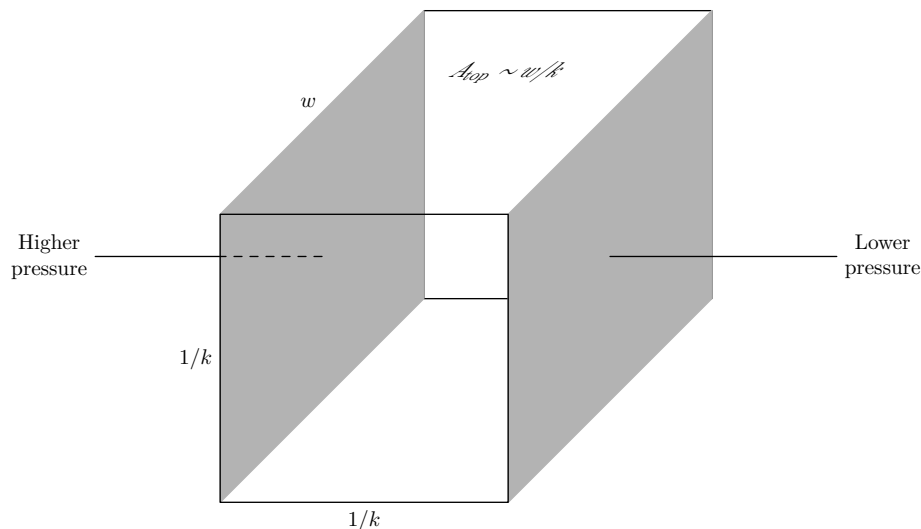


The higher pressure forces water underneath the crest to flow toward the trough, making the water level there rise. Like a swing sliding past equilibrium, the surface overshoots the equilibrium level to produce a new crest and the cycle repeats.

The next step is to quantify the model by estimating sizes, forces, speeds, and energies. In ?? we analyzed a messy mortality curve by replacing it with a more tractable shape: a rectangle. The method of discretization worked there, so try it again. ‘A method is a trick I use twice.’

—George Polyà. Water just underneath the surface moves quickly because of the pressure gradient. Farther down, it moves more slowly. Deep down it does not move at all. Replace this smooth falloff with a step function: Pretend that water down to a certain depth moves as a block, while deeper water stays still:

¹ When the reporters Woodward and Bernstein [2] were investigating criminal coverups during the Nixon administration, they received help from the mysterious ‘Deep Throat’, whose valuable advice was to ‘follow the money.’



How deep should this slab of water extend? By the Laplace-equation argument, the pressure variation falls off exponentially with depth, with length scale $1/k$. So assume that the slab has a similar length scale, that it has depth $1/k$. What choice do you have? On an infinitely deep ocean, the only length scale is $1/k$. How long should the slab be? Its length should be roughly the peak-to-trough distance of the wave because the surface height changes significantly over that distance. This distance is $1/k$. Actually, it is π/k (one-half of a period), but ignore constants. All the constants combine into a giant constant at the end, which dimensional analysis cannot determine anyway, so discard it now! The slab's width w is arbitrary and cancels by the end of any analysis.

So the slab of water has depth $1/k$, length $1/k$, and width w . Estimate the forces acting on it by estimating the pressure gradients. Across the width of the slab (the y direction), the water surface is level, so the pressure is constant along the width. Into the depths (the z direction), the pressure varies because of gravity – the ρgh term from hydrostatics – but that variation is just sufficient to prevent the slab from sinking. We care about only the pressure difference across the length, the direction that the wave moves. This pressure difference depends on the height of the crest, ξ and is $\Delta p \sim \rho g \xi$. This pressure difference acts on a cross-section with area $A \sim w/k$ to produce a force

$$F \sim \underbrace{w/k}_{\text{area}} \times \underbrace{\rho g \xi}_{\Delta p} = \rho g w \xi / k.$$

The slab has mass

$$m = \rho \times \underbrace{w/k^2}_{\text{volume}},$$

so the force produces an acceleration

$$a_{\text{slab}} \sim \underbrace{\frac{\rho g w \xi}{k}}_{\text{force}} / \underbrace{\frac{\rho w}{k^2}}_{\text{mass}} = g \xi k.$$

The factor of g says that the gravity produces the acceleration. Full gravitational acceleration is reduced by the dimensionless factor ξk , which is roughly the slope of the waves.

The acceleration of the slab determines the acceleration of the surface. If the slab moves a distance x , it sweeps out a volume of water $V \sim xA$. This water moves under the trough, and forces the surface upward a distance V/A_{top} . Because $A_{\text{top}} \sim A$ (both are $\sim w/k$), the surface moves the same distance x that the slab moves. Therefore, the slab's acceleration a_{slab} equals the acceleration a of the surface:

$$a \sim a_{\text{slab}} \sim g \xi k.$$

This equivalence of slab and surface acceleration does not hold in shallow water, where the bottom at depth h cuts off the slab before $1/k$; that story is told in [Section 10.2.12](#).

The slab argument is supposed to justify the deep-water dispersion relation derived by dimensional analysis. That relation contains frequency whereas acceleration relation does not. So massage it until ω appears. The acceleration relation contains a and ξ , whereas the dispersion relation does not. An alternative expression for the acceleration might make the acceleration relation more like the dispersion relation. With luck the expression will contain ω^2 , thereby producing the hoped-for ω^2 ; as a bonus, it will contain ξ to cancel the ξ in the acceleration relation.

In simple harmonic motion (springs!), acceleration is $a \sim \omega^2 \xi$, where ξ is the amplitude. In waves, which behave like springs, a is given by the same expression. Here's why. In time $\tau \sim 1/\omega$, the surface moves a distance $d \sim \xi$, so $a/\omega^2 \sim \xi$ and $a \sim \omega^2 \xi$. With this replacement, the acceleration relation becomes

$$\underbrace{\omega^2 \xi}_a \sim g \xi k,$$

or

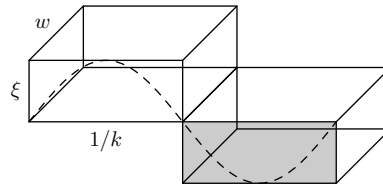
$$\omega^2 = 1 \times gk,$$

which is the longed-for dispersion relation with the correct dimensionless constant in red.

An exact calculation confirms the usual hope that the missing dimensionless constants are close to unity, or are unity. This fortune suggests that the procedures for choosing how to measure the lengths were reasonable. The derivation depended on two choices:

1. Replacing an exponentially falling variation in velocity potential by a step function with size equal to the length scale of the exponential decay.
2. Taking the length of the slab to be $1/k$ instead of π/k . This choice uses only 1 radian of the cycle as the characteristic length, instead of using a half cycle or π radians. Since 1 is a more natural dimensionless number than π is, choosing 1 radian rather than π or 2π radians often improves approximations.

Both approximations are usually accurate in order-of-magnitude calculations. Rarely, however, you will get caught by a factor of $(2\pi)^6$, and wish that you had used a full cycle instead of only 1 radian.



The derivation that resulted in the dispersion relation analyzed the motion of the slab using forces. Another derivation of it uses energy by balancing kinetic and potential energy. To make a wavy surface requires energy, as shown in the figure. The crest rises a characteristic height ξ above the zero of potential, which is the level surface. The volume of water moved upward is $\xi w/k$. So the potential energy is

$$PE_{\text{gravity}} \sim \underbrace{\rho \xi w/k}_{m} \times g \xi \sim \rho g w \xi^2 / k.$$

The kinetic energy is contained in the sideways motion of the slab and in the upward motion of the water pushed by the slab. The slab and surface move at the same speed; they also have the same acceleration. So the sideways and upward motions contribute similar energies. If you ignore

constants such as 2, you do not need to compute the energy contributed by both motions and can do the simpler computation, which is the sideways motion. The surface moves a distance ξ in a time $1/\omega$, so its velocity is $\omega\xi$. The slab has the same speed (except for constants) as the surface, so the slab's kinetic energy is

$$KE_{\text{deep}} \sim \underbrace{\rho w/k^2}_{m_{\text{slab}}} \times \underbrace{\omega^2 \xi^2}_{v^2} \sim \rho \omega^2 \xi^2 w/k^2.$$

This energy balances the potential energy

$$\underbrace{\rho \omega^2 \xi^2 w/k^2}_{KE} \sim \underbrace{\rho g w \xi^2/k}_{PE}.$$

Canceling the factor $\rho w \xi^2$ (in red) common to both energies leaves

$$\omega^2 \sim gk.$$

The energy method agrees with the force method, as it should, because energy can be derived from force by integration. The energy derivation gives an interpretation of the dimensionless group Π_2 :

$$\Pi_2 \sim \frac{\text{kinetic energy in slab}}{\text{gravitational potential energy}} \sim \frac{\omega^2}{gk}.$$

The gravity-wave dispersion relation $\omega^2 = gk$ is equivalent to $\Pi_2 \sim 1$, or to the assertion that kinetic and gravitational potential energy are comparable in wave motion. This rough equality is no surprise because waves are like springs. In spring motion, kinetic and potential energies have equal averages, a consequence of the virial theorem.

The dispersion relation was derived in three ways: by dimensional analysis, energy, and force. Using multiple methods increases our confidence not only in the result but also in the methods. 'I have said it thrice: What I tell you three times is true.'

—Lewis Carroll, *Hunting of the Snark*.

We gain confidence in the methods of dimensional analysis and in the slab model for waves. If we study nonlinear waves, for example, where the wave height is no longer infinitesimal, we can use the same techniques along with the slab model with more confidence.

With reasonable confidence in the dispersion relation, it's time study its consequences: the phase and group velocities. The crests move at the phase velocity: $v_{ph} = \omega/k$. For deep-water gravity waves, this velocity becomes

$$v_{ph} = \sqrt{\frac{g}{k}},$$

or, using the dispersion relation to replace k by ω ,

$$v_{ph} = \frac{g}{\omega}.$$

Let's check upstairs and downstairs. Who knows where ω belongs, but g drives the waves so it should and does live upstairs.

In an infinite, single-frequency wave train, the crests and troughs move at the phase speed. However, a finite wave train contains a mixture of frequencies, and the various frequencies move at different speeds as given by

$$v_{ph} = \frac{g}{\omega}.$$

Deep water is **dispersive**. Dispersion makes a finite wave train travel with the group velocity, given by $v_g = \partial\omega/\partial k$, as explained in [Section 10.2.2](#). The group velocity is

$$v_g = \frac{\partial}{\partial k} \sqrt{gk} = \frac{1}{2} \sqrt{\frac{g}{k}} = \frac{1}{2} v_{ph}.$$

So the group velocity is one-half of the phase velocity, as the result for power-law dispersion relation predicts. Within a wave train, the crests move at the phase velocity, twice the group velocity, shrinking and growing to fit under the slower-moving envelope.

An everyday consequence is that ship wakes trail the ship. A ship moving with velocity v creates gravity waves with $v_{ph} = v$. The waves combine to produce wave trains that propagate forward with the group velocity, which is only $v_{ph}/2 = v/2$. From the ship's point of view, these gravity waves travel backward. In fact, they form a wedge, and the opening angle of the wedge depends on the one-half that arises from the exponent.

10.2.6 Surfing

Let's apply the dispersion relation to surfing. Following one winter storm reported in the *Los Angeles Times* – the kind of storm that brings cries of 'Surf's up!' – waves arrived at Los Angeles beaches roughly every 18s. How fast were the storm winds that generated the waves? Wind pushes the crests as long as they move more slowly than the wind. After a long-enough push, the crests move with nearly the wind speed. Therefore

the phase velocity of the waves is an accurate approximation to the wind speed.

The phase velocity is g/ω . In terms of the wave period T , this velocity is $v_{ph} = gT/2\pi$, so

$$v_{wind} \sim v_{ph} \sim \frac{\overbrace{10 \text{ m s}^{-2}}^g \times \overbrace{18 \text{ s}}^T}{2 \times 3} \sim 30 \text{ m s}^{-1}.$$

In units more familiar to Americans, this wind speed is 60 mph, which is a strong storm: about 10 on the Beaufort wind scale ('whole gale/storm'). The wavelength is given by

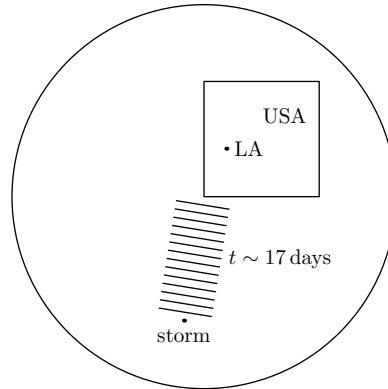
$$\lambda = v_{ph}T \sim 30 \text{ m s}^{-1} \times 18 \text{ s} \sim 500 \text{ m}.$$

On the open ocean, the crests are separated by half a kilometer. Near shore they bunch up because they feel the bottom; this bunching is a consequence of the shallow-water dispersion relation, the topic of [Section 10.2.13](#).

In this same storm, the waves arrived at 17 s intervals the following day: a small decrease in the period. Before racing for the equations, first check that this decrease in period is reasonable. This precaution is a sanity check. If the theory is wrong about a physical effect as fundamental as a sign – whether the period should decrease or increase – then it neglects important physics. The storm winds generate waves of different wavelengths and periods, and the different wavelengths sort themselves during the trip from the far ocean to Los Angeles. Group and phase velocity are proportional to $1/\omega$, which is proportional to the period. So longer-period waves move faster, and the 18 s waves should arrive before the 17 s waves. They did! The decline in the interval allows us to calculate the distance to the storm. In their long journey, the 18 s waves raced 1 day ahead of the 17 s waves. The ratio of their group velocities is

$$\frac{\text{velocity}(18 \text{ s waves})}{\text{velocity}(17 \text{ s waves})} = \frac{18}{17} = 1 + \frac{1}{17}.$$

so the race must have lasted roughly $t \sim 17 \text{ days} \sim 1.5 \cdot 10^6 \text{ s}$. The wave train moves at the group velocity, $v_g = v_{ph}/2 \sim 15 \text{ m s}^{-1}$, so the storm distance was $d \sim tv_g \sim 2 \cdot 10^4 \text{ km}$, or roughly halfway around the world, an amazingly long and dissipation-free journey.



10.2.7 Speedboating

Our next application of the dispersion relation is to speedboating: How fast can a boat travel? We exclude hydroplaning boats from our analysis (even though some speedboats can hydroplane). Longer boats generally move faster than shorter boats, so it is likely that the length of the boat, l , determines the top speed. The density of water might matter. However, v (the speed), ρ , and l cannot form a dimensionless group. So look for another variable. Viscosity is irrelevant because the Reynolds number for boat travel is gigantic. Even for a small boat of length 5 m, creeping along at 2 m s^{-1} ,

$$\text{Re} \sim \frac{500 \text{ cm} \times 200 \text{ cm s}^{-1}}{10^{-2} \text{ cm}^2 \text{ s}^{-1}} \sim 10^7.$$

At such a huge Reynolds number, the flow is turbulent and nearly independent of viscosity ([Section 7.3.7](#)). Surface tension is also irrelevant, because boats are much longer than a ripple wavelength (roughly 1 cm). The search for new variables is not meeting with success. Perhaps gravity is relevant. The four variables v , ρ , g , and l , build from three dimensions, produce one dimensionless group: v^2/gl , also called the **Froude number**:

$$\text{Fr} \equiv \frac{v^2}{gl}.$$

The critical Froude number, which determines the maximum boat speed, is a dimensionless constant. As usual, we assume that the constant is unity. Then the maximum boating speed is:

$$v \sim \sqrt{gl}.$$

A rabbit has jumped out of our hat. What physical mechanism justifies this dimensional-analysis result? Follow the waves as a boat plows through water. The moving boat generates waves (the wake), and it rides on one of those waves. Take the bow wave: It is a gravity wave with $v_{ph} \sim v_{boat}$. Because $v_{ph}^2 = \omega^2/k^2$, the dispersion relation tells us that

$$v_{boat}^2 \sim \frac{\omega^2}{k^2} = \frac{g}{k} = g\lambda,$$

where $\lambda \equiv 1/k = \lambda/2\pi$. So the wavelength of the waves is roughly v_{boat}^2/g . The other length in this problem is the boat length; so the Froude number has this interpretation:

$$Fr = \frac{v_{boat}^2/g}{l} \sim \frac{\text{wavelength of bow wave}}{\text{length of boat}}.$$

Why is $Fr \sim 1$ the critical number, the assumption in finding the maximum boat speed? Interesting and often difficult physics occurs when a dimensionless number is near unity. In this case, the physics is as follows. The wave height changes significantly in a distance λ ; if the boat's length l is comparable to λ , then the boat rides on its own wave and tilts upward. Tilting upward, it presents a large cross-section to the water, and the drag becomes huge. [Catamarans and hydrofoils skim the water, so this kind of drag does not limit their speed. The hydrofoil makes a much quicker trip across the English channel than the ferry makes, even though the hydrofoil is much shorter.] So the top speed is given by

$$v_{boat} \sim \sqrt{gl}.$$

For a small motorboat, with length $l \sim 5$ m, this speed is roughly 7 m s^{-1} , or 15 mph. Boats (for example police boats) do go faster than the nominal top speed, but it takes plenty of power to fight the drag, which is why police boats have huge engines.

The Froude number in surprising places. It determines, for example, the speed at which an animal's gait changes from a walk to a trot or, for animals that do not trot, to a run. In [Section 10.2.7](#) it determines maximum boating speed. The Froude number is a ratio of potential energy to kinetic energy, as massaging the Froude number shows:

$$Fr = \frac{v^2}{gl} = \frac{\textcolor{red}{m}v^2}{\textcolor{red}{m}gl} \sim \frac{\text{kinetic energy}}{\text{potential energy}}.$$

Here the massage technique was multiplication by unity (in red). In this example, the length l is a horizontal length, so gl is not a gravitational

energy, but it has a similar structure and in other examples often has an easy interpretation as gravitational energy.

10.2.8 Walking

In the Froude number for walking speed, l is leg length, and gl is a potential energy. For a human with leg length $l \sim 1$ m, the condition $Fr \sim 1$ implies that $v \sim 3 \text{ m s}^{-1}$ or 6 mph. This speed is a rough estimate for the top speed for a race walker. The world record for men's race walking was once held by Bernado Segura of Mexico. He walked 20 km in 1h:17m:25.6s, for a speed of 4.31 m s^{-1} .

This example concludes the study of gravity waves on deep water, which is one corner of the world of waves.

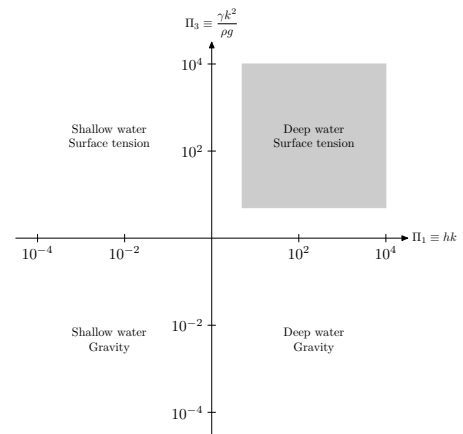
10.2.9 Ripples on deep water

For small wavelengths (large k), surface tension rather than gravity provides the restoring force. This choice brings us to the shaded corner of the figure. If surface tension rather than gravity provides the restoring force, then g vanishes from the final dispersion relation. How to get rid of g and find the new dispersion relation? You could follow the same pattern as for gravity waves ([Section 10.2.5](#)). In that situation, the surface tension γ was irrelevant, so we discarded the group $\Pi_3 \equiv \gamma k^2 / \rho g$. Here, with g irrelevant you might try the same trick: Π_3 contains g so discard it.

In that argument lies infanticide, because it also throws out the physical effect that determines the restoring force, namely surface tension. To retrieve the baby from the bathwater, you cannot throw out $\gamma k^2 / \rho g$ directly. Instead you have to choose the form of the dimensionless function f_{deep} in so that only gravity vanishes from the dispersion relation.

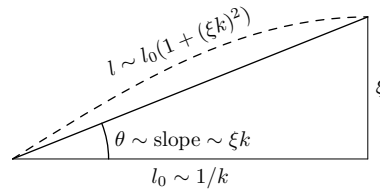
The deep-water dispersion relation contains one power of g in front. The argument of f_{deep} also contains one power of g , in the denominator. If f_{deep} has the form $f_{\text{deep}}(x) \sim x$, then g cancels. With this choice, the dispersion relation is

$$\omega^2 = 1 \times \frac{\gamma k^3}{\rho}.$$



Again the dimensionless constant from exact calculation (in red) is unity, which we would have assumed anyway. Let's reuse the slab argument to derive this relation.

In the slab picture, replace gravitational by surface-tension energy, and again balance potential and kinetic energies. The surface of the water is like a rubber sheet. A wave disturbs the surface and stretches the sheet. This stretching creates area ΔA and therefore requires energy $\gamma \Delta A$. So to estimate the energy, estimate the extra area that a wave of amplitude ξ and wavenumber k creates. The extra area depends on the extra length in a sine wave compared to a flat line. The typical slope in the sine wave $\xi \sin kx$ is ξk . Instead of integrating to find the arc length, you can approximate the curve as a straight line with slope ξk :



Relative to the level line, the tilted line is longer by a factor $1 + (\xi k)^2$.

As before, imagine a piece of a wave, with characteristic length $1/k$ in the x direction and width w in the y direction. The extra area is

$$\Delta A \sim \underbrace{w/k}_{\text{level area}} \times \underbrace{(\xi k)^2}_{\text{fractional increase}} \sim w \xi^2 k.$$

The potential energy stored in this extra surface is

$$PE_{\text{ripple}} \sim \gamma \Delta A \sim \gamma w \xi^2 k.$$

The kinetic energy in the slab is the same as it is for gravity waves, which is:

$$KE \sim \rho \omega^2 \xi^2 w / k^2.$$

Balancing the energies

$$\underbrace{\rho \omega^2 \xi^2 w / k^2}_{KE} \sim \underbrace{\gamma w \xi^2 k}_{PE},$$

gives

$$\omega^2 \sim \gamma k^3 / \rho.$$

This dispersion relation agrees with the result from dimensional analysis. For deep-water gravity waves, we used both energy and force arguments to re-derive the dispersion relation. For ripples, we worked out the energy argument, and you are invited to work out the corresponding force argument.

The energy calculation completes the interpretations of the three dimensionless groups. Two are already done: Π_1 is the dimensionless depth and Π_2 is ratio of kinetic energy to gravitational potential energy. We constructed Π_3 as a group that compares the effects of surface tension and gravity. Using the potential energy for gravity waves and for ripples, the comparison becomes more precise:

$$\begin{aligned}\Pi_3 &\sim \frac{\text{potential energy in a ripple}}{\text{potential energy in a gravity wave}} \\ &\sim \frac{\gamma w \xi^2 k}{\rho g w \xi^2 / k} \\ &\sim \frac{\gamma k^2}{\rho g}.\end{aligned}$$

Alternatively, Π_3 compares $\gamma k^2 / \rho$ with g :

$$\Pi_3 \equiv \frac{\gamma k^2 / \rho}{g}.$$

This form of Π_3 may seem like a trivial revision of $\gamma k^2 / \rho g$. However, it suggests an interpretation of surface tension: that surface tension acts like an effective gravitational field with strength

$$g_{\text{surface tension}} = \gamma k^2 / \rho,$$

In a balloon, the surface tension of the rubber implies a higher pressure inside than outside. Similarly in wave the water skin implies a higher pressure underneath the crest, which is curved like a balloon; and a lower pressure under the trough, which is curved opposite to a balloon. This pressure difference is just what a gravitational field with strength $g_{\text{surface tension}}$ would produce. This trick of effective gravity, which we used for the buoyant force on a falling marble ([Section 7.3.4](#)), is now promoted to a method (a trick used twice).

So replace g in the gravity-wave potential energy with this effective g to get the ripple potential energy:

$$\underbrace{\rho g w \xi^2 / k}_{PE(\text{gravity wave})} \xrightarrow{g \rightarrow \gamma k^2 / \rho} \underbrace{\gamma w \xi^2 k}_{PE(\text{ripple})}.$$

The left side becomes the right side after making the substitution above the arrow. The same replacement in the gravity-wave dispersion relation produces the ripple dispersion relation:

$$\omega^2 = gk \xrightarrow{g \rightarrow \gamma k^2 / \rho} \omega^2 = \frac{\gamma k^3}{\rho}.$$

The interpretation of surface tension as effective gravity is useful when we combine our solutions for gravity waves and for ripples, in [Section 10.2.11](#) and [Section 10.2.16](#). Surface tension and gravity are symmetric: We could have reversed the analysis and interpreted gravity as effective surface tension. However, gravity is the more familiar force, so we use effective gravity rather than effective surface tension.

With the dispersion relation you can harvest the phase and group velocities. The phase velocity is

$$v_{ph} \equiv \frac{\omega}{k} = \sqrt{\frac{\gamma k}{\rho}},$$

and the group velocity is

$$v_g \equiv \frac{\partial \omega}{\partial k} = \frac{3}{2} v_{ph}.$$

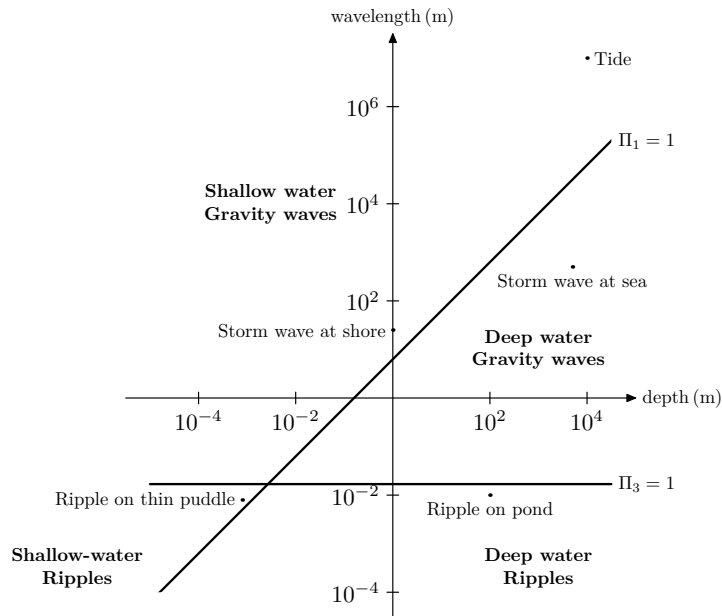
The factor of 3/2 is a consequence of the form of the dispersion relation: $\omega \propto k^{3/2}$; for gravity waves, $\omega \propto k^{1/2}$, and the corresponding factor is 1/2. In contrast to deep-water waves, a train of ripples moves *faster* than the phase velocity. So, ripples steam ahead of a boat, whereas gravity waves trail behind.

10.2.10 Typical ripples

Let's work out speeds for typical ripples, such as the ripples from dropping a pebble into a pond. From observation, these ripples have wavelength $\lambda \sim 1$ cm, and therefore wavenumber $k = 2\pi/\lambda \sim 6 \text{ cm}^{-1}$. The surface tension of water (??) is $\gamma \sim 0.07 \text{ J m}^{-2}$. So the phase velocity is

$$v_{ph} = \left(\frac{\overbrace{0.07 \text{ J m}^{-2}}^{\gamma} \times \overbrace{600 \text{ m}^{-1}}^k}{\underbrace{10^3 \text{ kg m}^{-3}}_{\rho}} \right)^{1/2} \sim 21 \text{ cm s}^{-1}.$$

According to relation between phase and group velocities, the group velocity is 50 percent larger than the phase velocity: $v_g \sim 30 \text{ cm s}^{-1}$. This wavelength of 1 cm is roughly the longest wavelength that still qualifies as a ripple, as shown in an earlier figure repeated here:



The third dimensionless group, which distinguishes ripples from gravity waves, has value

$$\Pi_3 \equiv \frac{\gamma k^2}{\rho g} \sim \frac{\overbrace{0.07 \text{ J m}^{-2}}^{\gamma} \times \overbrace{3.6 \cdot 10^5 \text{ m}^{-2}}^{k^2}}{\underbrace{10^3 \text{ kg m}^{-3}}_{\rho} \times \underbrace{10 \text{ m s}^{-2}}_g} \sim 2.6.$$

With a slightly smaller k , the value of Π_3 would slide into the gray zone $\Pi_3 \approx 1$. If k were yet smaller, the waves would be gravity waves. Other ripples, with a larger k , have a shorter wavelength, and therefore move faster: 21 cm s^{-1} is roughly the minimum phase velocity for ripples. This minimum speed explains why we see mostly $\lambda \sim 1 \text{ cm}$ ripples when we

drop a pebble in a pond. The pebble excites ripples of various wavelengths; the shorter ones propagate faster and the 1 cm ones straggle, so we see the stragglers clearly, without admixture of other ripples.

10.2.11 Combining ripples and gravity waves on deep water

With two corners assembled – gravity waves and ripples in deep water – you can connect the corners to form the deep-water edge. The dispersion relations, for convenience restated here, are

$$\omega^2 = \begin{cases} gk, & \text{gravity waves;} \\ \gamma k^3/\rho, & \text{ripples.} \end{cases}$$

With a little courage, you can combine the relations in these two extreme regimes to produce a dispersion relation valid for gravity waves, for ripples, and for waves in between.

Both functional forms came from the same physical argument of balancing kinetic and potential energies. The difference was the source of the potential energy: gravity or surface tension. On the top half of the world of waves, surface tension dominates gravity; on the bottom half, gravity dominates surface tension. Perhaps in the intermediate region, the two contributions to the potential energy simply add. If so, the combination dispersion relation is the sum of the two extremes:

$$\omega^2 = gk + \gamma k^3/\rho.$$

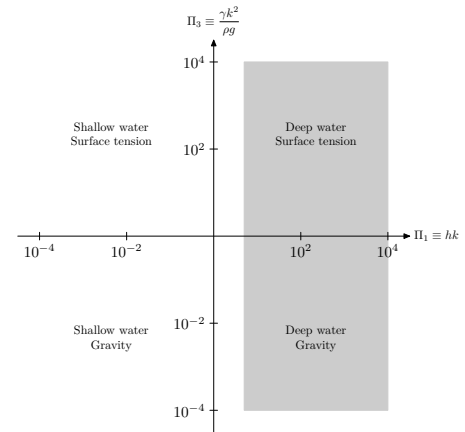
This result is exact (which is why we used an equality). When in doubt, try the simplest solution.

You can increase your confidence in this result by using the effective gravity produced by surface tension. The two sources of gravity – real and effective – simply add, to make

$$g_{\text{total}} = g + g_{\text{surface tension}} = g + \frac{\gamma k^2}{\rho}.$$

Replace g by g_{total} in $\omega^2 = gk$ reproduces the deep-water dispersion relation:

$$\omega^2 = \left(g + \frac{\gamma k^2}{\rho} \right) k = gk + \gamma k^3/\rho.$$



This dispersion relation tells us wave speeds for all wavelengths or wavenumbers. The phase velocity is

$$v_{ph} \equiv \frac{\omega}{k} = \sqrt{\frac{\gamma k}{\rho} + \frac{g}{k}}.$$

Let's check upstairs and downstairs. Surface tension and gravity drive the waves, so γ and g should be upstairs. Inertia slows the waves, so ρ should be downstairs. The phase velocity passes these tests.

As a function of wavenumber, the two terms in the square root compete to increase the speed. The surface-tension term wins at high wavenumber; the gravity term wins at low wavenumber. So there is an intermediate, minimum-speed wavenumber, k_0 , which we can estimate by balancing the surface tension and gravity contributions:

$$\frac{\gamma k_0}{\rho} \sim \frac{g}{k_0}.$$

This computation is an example of order-of-magnitude minimization. The minimum-speed wavenumber is

$$k_0 \sim \sqrt{\frac{\rho g}{\gamma}}.$$

Interestingly, $1/k_0$ is the maximum size of raindrops. At this wavenumber $\Pi_3 = 1$: These waves lie just on the border between ripples and gravity waves. Their phase speed is

$$v_0 \sim \sqrt{\frac{2g}{k_0}} \sim \left(\frac{4\gamma g}{\rho}\right)^{1/4}.$$

In water, the critical wavenumber is $k_0 \sim 4 \text{ cm}^{-1}$, so the critical wavelength is $\lambda_0 \sim 1.5 \text{ cm}$; the speed is

$$v_0 \sim 23 \text{ cm s}^{-1}.$$

We derived the speed dishonestly. Instead of using the maximum–minimum methods of calculus, we balanced the two contributions. A calculus derivation confirms the minimum phase velocity. A tedious calculus calculation shows that the minimum group velocity is

$$v_g \approx 17.7 \text{ cm s}^{-1}.$$

[If you try to reproduce this calculation, be careful because the minimum group velocity is not the group velocity at k_0 .]

Let's do the minimizations honestly. The calculation is not too messy if it's done with good formula hygiene plus a useful diagram, and the proper method is useful in many physical maximum–minimum problems. We illustrate the methods by finding the minimum of the phase velocity. That equation contains constants – ρ , γ , and g – which carry through all the differentiations. To simplify the manipulations, choose a convenient set of units in which

$$\rho = \gamma = g = 1.$$

The analysis of waves uses three basic dimensions: mass, length, and time. Choosing three constants equal to unity uses up all the freedom. It is equivalent to choosing a canonical mass, length, and time, and thereby making all quantities dimensionless. Don't worry: The constants will return at the end of the minimization.

In addition to constants, the phase velocity also contains a square root. As a first step in formula hygiene, minimize instead v_{ph}^2 . In the convenient unit system, it is

$$v_{ph}^2 = k + \frac{1}{k}.$$

This minimization does not need calculus, even to do it exactly. The two terms are both positive, so you can use the arithmetic-mean–geometric-mean inequality (affectionately known as AM–GM) for k and $1/k$. The inequality states that, for positive a and b ,

$$\underbrace{(a + b)/2}_{AM} \geq \underbrace{\sqrt{ab}}_{GM},$$

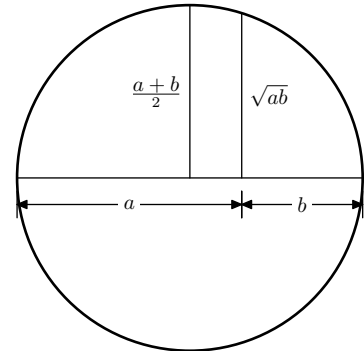
with equality when $a = b$.

The figure shows a geometric proof of this inequality. You are invited to convince yourself that the figure is a proof. With $a = k$ and $b = 1/k$ the geometric mean is unity, so the arithmetic mean is ≥ 1 . Therefore

$$k + \frac{1}{k} \geq 2,$$

with equality when $k = 1/k$, namely when $k = 1$. At this wavenumber the phase velocity is $\sqrt{2}$. Still in this unit system, the dispersion relation is

$$\omega = \sqrt{k^3 + k},$$



and the group velocity is

$$v_g = \frac{\partial}{\partial k} \sqrt{k^3 + k},$$

which is

$$v_g = \frac{1}{2} \frac{3k^2 + 1}{\sqrt{k^3 + k}}.$$

At $k = 1$ the group velocity is also $\sqrt{2}$: These borderline waves have equal phase and group velocity. This equality is reasonable. In the gravity-wave regime, the phase velocity is greater than the group velocity. In the ripple regime, the phase velocity is less than the group velocity. So they must be equal somewhere in the intermediate regime.

To convert $k = 1$ back to normal units, multiply it by unity in the form of a convenient product of ρ , γ , and g (which are each equal to 1 for the moment). How do you make a length from ρ , γ , and g ? The form of the result says that $\sqrt{\rho g / \gamma}$ has units of L^{-1} . So $k = 1$ really means $k = 1 \times \sqrt{\rho g / \gamma}$, which is the same as the order-of-magnitude minimization. This exact calculation shows that the missing dimensionless constant is 1.

The minimum group velocity is more complicated than the minimum phase velocity because it requires yet another derivative. Again, remove the square root and minimize v_g^2 . The derivative is

$$\frac{\partial}{\partial k} \underbrace{\frac{9k^4 + 6k^2 + 1}{k^3 + k}}_{v_g^2} = \frac{(3k^2 + 1)(3k^4 + 6k^2 - 1)}{(k^3 + k)^2}.$$

Equating this derivative to zero gives $3k^4 + 6k^2 - 1 = 0$, which is a quadratic in k^2 , and has positive solution

$$k_1 = \sqrt{-1 + \sqrt{4/3}} \sim 0.393.$$

At this k , the group velocity is

$$v_g(k_1) \approx 1.086.$$

In more usual units, this minimum velocity is

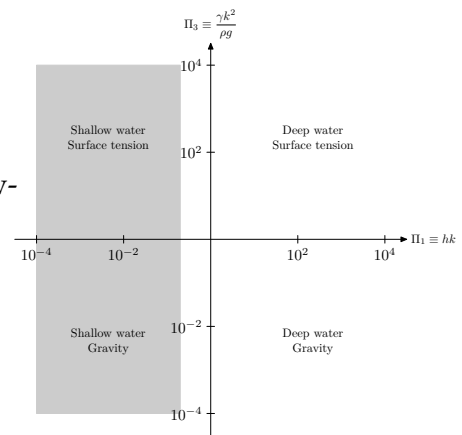
$$v_g \approx 1.086 \left(\frac{\gamma g}{\rho} \right)^{1/4}.$$

With the density and surface tension of water, the minimum group velocity is 17.7 cm s^{-1} , as claimed previously.

After dropping a pebble in a pond, you see a still circle surrounding the drop point. Then the circle expands at the minimum group velocity given. Without a handy pond, try the experiment in your kitchen sink: Fill it with water and drop in a coin or a marble. The existence of a minimum phase velocity, is useful for bugs that walk on water. If they move slower than 23 cm s^{-1} , they generate no waves, which reduces the energy cost of walking.

10.2.12 Shallow water

In shallow water, the height h , absent in the deep-water calculations, returns to complicate the set of relevant variables. We are now in the shaded region of the figure. This extra length scale gives too much freedom. Dimensional analysis alone cannot deduce the shallow-water form of the magic function f in the dispersion relation. The slab argument can do the job, but it needs a few modifications for the new physical situation.



In deep water the slab has depth $1/k$. In shallow water, however, where $h \ll 1/k$, the bottom of the ocean arrives before that depth. So the shallow-water slab has depth h . Its length is still $1/k$, and its width is still w . Because the depth changed, the argument about how the water flows is slightly different. In deep water, where the slab has depth equal to length, the slab and surface move the same distance. In shallow water, with a slab thinner by hk , the surface moves more slowly than the slab because less water is being moved around. It moves more slowly by the factor hk . With wave height ξ and frequency ω , the surface moves with velocity $\xi\omega$, so the slab moves (sideways) with velocity $v_{\text{slab}} \sim \xi\omega/hk$. The kinetic energy in the water is contained mostly in the slab, because the upward motion is much slower than the slab motion. This energy is

$$KE_{\text{shallow}} \sim \underbrace{\rho w h / k}_{\text{mass}} \times \underbrace{(\xi\omega/hk)^2}_{v^2} \sim \frac{\rho w \xi^2 \omega^2}{hk^3}.$$

This energy balances the potential energy, a computation we do for the two limiting cases: ripples and gravity waves.

10.2.13 Gravity waves on shallow water

We first specialize to gravity waves – the shaded region in the figure – where water is shallow and wavelengths are long. These conditions include tidal waves, waves generated by undersea earthquakes, and waves approaching a beach. For gravity waves, the potential energy is

$$PE \sim \rho g w \xi^2 / k.$$

This energy came from the distortion of the surface, and it is the same in shallow water (as long as the wave amplitude is small compared with the depth and wavelength). [The dominant force (gravity or surface tension) determines the potential energy. As we see when we study shallow-water ripples, in [Section 10.2.15](#), the water depth determines the kinetic energy.]

Balancing this energy against the kinetic energy gives:

$$\underbrace{\frac{\rho w \xi^2 \omega^2}{h k^3}}_{KE} \sim \underbrace{\rho g w \xi^2 / k}_{PE}.$$

So

$$\omega^2 = 1 \times g h k^2.$$

Once again, the correct, honestly calculated dimensionless constant (in red) is unity. So, for gravity waves on shallow water, the function f has the form

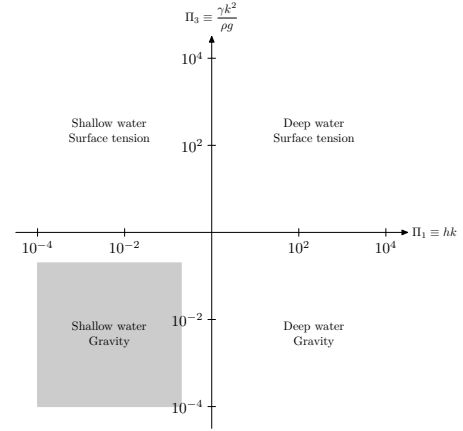
$$f_{\text{shallow}}(kh, \frac{\gamma k^2}{\rho g}) = kh.$$

Since $\omega \propto k^1$, the group and phase velocities are equal and independent of frequency:

$$v_{ph} = \frac{\omega}{k} = \sqrt{gh},$$

$$v_g = \frac{\partial \omega}{\partial k} = \sqrt{gh}.$$

Shallow water is **nondispersive**: All frequencies move at the same velocity, so pulses composed of various frequencies propagate without smearing.



10.2.14 Tidal waves

Undersea earthquakes illustrate the danger in such unity. If an earthquake strikes off the coast of Chile, dropping the seafloor, it generates a shallow-water wave. This wave travels without distortion to Japan. The wave speed is $v \sim \sqrt{4000 \text{ m} \times 10 \text{ m s}^{-2}} \sim 200 \text{ m s}^{-1}$: The wave can cross a 10^4 km ocean in half a day. As it approaches shore, where the depth decreases, the wave slows, grows in amplitude, and becomes a large, destructive wave hitting land.

10.2.15 Ripples on shallow water

Ripples on shallow water – the shaded region in the figure – are rare. They occur when raindrops land in a shallow rain puddle, one whose depth is less than 1 mm. Even then, only the longest-wavelength ripples, where $\lambda \sim 1 \text{ cm}$, can feel the bottom of the puddle (the requirement for the wave to be a shallow-water wave). The potential energy of the surface is given by

$$PE_{\text{ripple}} \sim \gamma \Delta A \sim \gamma w \xi^2 k.$$

Although that formula applied to deep water, the water depth does not affect the potential energy, so we can use the same formula for shallow water.

The dominant force – here, surface tension – determines the potential energy. Balancing the potential energy and the kinetic energy gives:

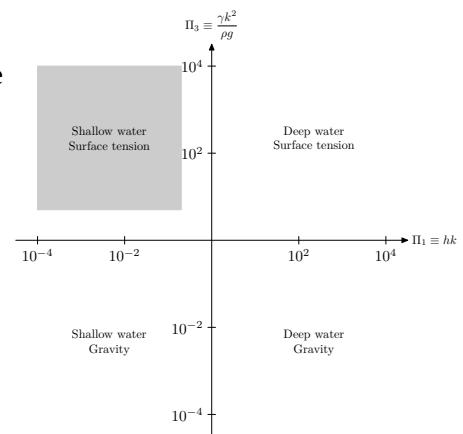
$$\underbrace{\frac{\rho w \xi^2 \omega^2}{h k^3}}_{KE} \sim \underbrace{\frac{w}{k} \gamma (k \xi)^2}_{PE}.$$

Then

$$\omega^2 \sim \frac{\gamma h k^4}{\rho}.$$

The phase velocity is

$$v_{ph} = \frac{\omega}{k} = \sqrt{\frac{\gamma h k^2}{\rho}},$$



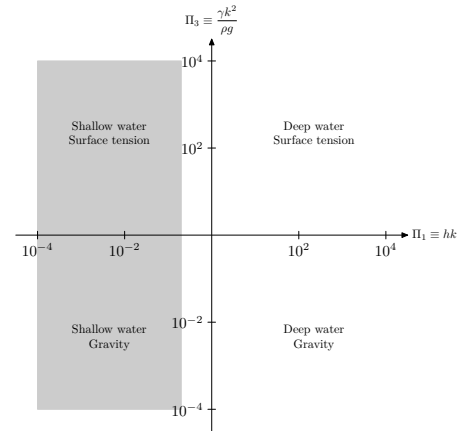
and the group velocity is $v_g = 2v_{ph}$ (the form of the dispersion relation is $\omega \propto k^2$). For $h \sim 1$ mm, this speed is

$$v \sim \left(\frac{0.07 \text{ N m}^{-1} \times 10^{-3} \text{ m} \times 3.6 \cdot 10^5 \text{ m}^{-2}}{10 \text{ kg m}^{-3}} \right)^{1/2} \sim 16 \text{ cm s}^{-1}.$$

10.2.16 Combining ripples and gravity waves on shallow water

This result finishes the last two corners of the world of waves: shallow-water ripples and gravity waves. Connect the corners to make an edge by studying general shallow-water waves. This region of the world of waves is shaded in the figure. You can combine the dispersion relations for ripples with that for gravity waves using two equivalent methods. Either add the two extreme-case dispersion relations or use the effective gravitational field in the gravity-wave dispersion relation. Either method produces

$$\omega^2 \sim k^2 \left(gh + \frac{\gamma h k^2}{\rho} \right).$$

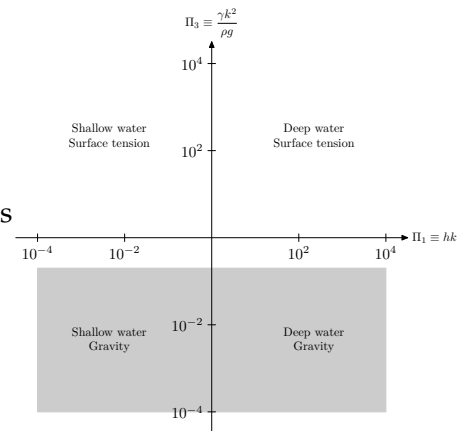


10.2.17 Combining deep- and shallow-water gravity waves

Now examine the gravity-wave edge of the world, shaded in the figure. The deep- and shallow-water dispersion relations are:

$$\omega^2 = gk \times \begin{cases} 1, & \text{deep water;} \\ hk, & \text{shallow water.} \end{cases}$$

To interpolate between the two regimes requires a function $f(hk)$ that asymptotes to 1 as $hk \rightarrow \infty$ and to hk as $hk \rightarrow 0$. Arguments based on guessing functional forms have an honored history in physics. Planck derived the black-body spectrum by interpolating between the high- and low-frequency limits of what was known at the time. We are not deriving quantum mechanics, but the principle is the same: In new areas, whether new to you or new to everyone,



you need a bit of courage. One simple interpolating function is $\tanh kh$. Then the one true gravity wave dispersion relation is:

$$\omega^2 = gk \tanh kh.$$

This educated guess is plausible because $\tanh kh$ falls off exponentially as $h \rightarrow \infty$, in agreement with the argument based on Laplace's equation. In fact, this guess is correct.

10.2.18 Combining deep- and shallow-water ripples

We now examine the final edge: ripples in shallow and deep water, as shown in the figure. In [Section 10.2.17](#), $\tanh kh$ interpolated between kh and 1 as kh went from 0 to ∞ (as the water went from shallow to deep). Probably the same trick works for ripples, because the Laplace-equation argument, which justified the $\tanh kh$, does not depend on the restoring force. The relevant dispersion relations:

$$\omega^2 = \begin{cases} \gamma k^3 / \rho, & \text{if } kh \gg 1; \\ \gamma h k^4 / \rho, & \text{if } kh \ll 1. \end{cases}$$

If we factor out $\gamma k^3 / \rho$, the necessary transformation becomes clear:

$$\omega^2 = \frac{\gamma k^3}{\rho} \times \begin{cases} 1, & \text{if } kh \gg 1; \\ kh, & \text{if } kh \ll 1. \end{cases}$$

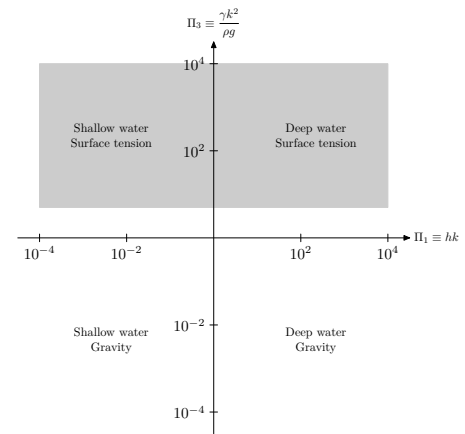
This ripple result looks similar to the gravity-wave result, so make the same replacement:

$$\begin{cases} 1, & \text{if } kh \gg 1, \\ kh, & \text{if } kh \ll 1, \end{cases} \text{ becomes } \tanh kh.$$

Then you get the general ripple dispersion relation:

$$\omega^2 = \frac{\gamma k^3}{\rho} \tanh kh.$$

This dispersion relation does not have much practical interest because, at the cost of greater complexity than the deep-water ripple dispersion relation, it adds coverage of only a rare case: ripples on ponds. We include it

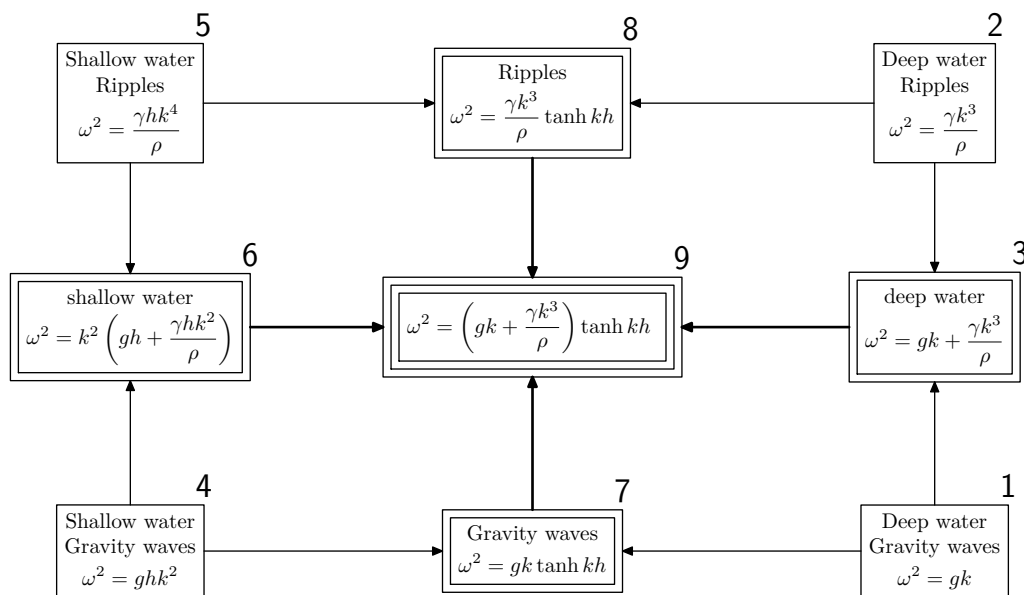


for completeness, to visit all four edges of the world, in preparation for the grand combination coming up next.

10.2.19 Combining all the analyses

Now we can replace g with g_{total} , to find the One True Dispersion Relation:

$$\omega^2 = (gk + \gamma k^3 / \rho) \tanh kh.$$



Each box in the figure represents a special case. The numbers next to the boxes mark the order in which we studied that limit. In the final step (9), we combined all the analyses into the superbox in the center, which contains the dispersion relation for all waves: gravity waves or ripples, shallow water or deep water. The arrows show how we combined smaller, more specialized corner boxes into the more general edge boxes (double ruled), and the edge regions into the universal center box (triple ruled).

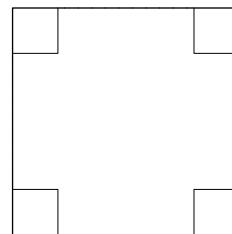
In summary, we studied water waves by investigating dispersion relations. We mapped the world of waves, explored the corners and then the edges, and assembled the pieces to form an understanding of the complex, complete solution. The whole puzzle, solved, is shown in the figure. Considering limiting cases and stitching them together makes the analysis tractable and comprehensible.

10.2.20 What you have learned

1. *Phase and group velocities.* Phase velocity says how fast crests in a single wave move. In a packet of waves (several waves added together), group velocity is the phase velocity of the envelope.
2. *Discretize.* A complicated functional relationship, such as a dispersion relation, is easier to understand in a discrete limit: for example, one that allows only two (ω, k) combinations. This discretization helped explain the meaning of group velocity.
3. *Four regimes.* The four regimes of wave behavior are characterized by two dimensionless groups: a dimensionless depth and a dimensionless ratio of surface tension to gravitational energy.
4. *Look for springs.* Look for springs when a problem has kinetic- and potential-energy reservoirs and energy oscillates between them. A key characteristic of spring motion is overshoot: The system must zoom past the equilibrium configuration of zero potential energy.
5. *Most missing constants are unity.* In analyses of waves and springs, the missing dimensionless constants are usually unity. This fortunate result comes from the virial theorem, which says that the average potential and kinetic energies are equal for a $F \propto r$ force (a spring force). So balancing the two energies is exact in this case.
6. *Minimum speed.* Objects moving below a certain speed (in deep water) generate no waves. This minimum speed is the result of cooperation between gravity and surface tension. Gravity keeps long-wavelength waves moving quickly. Surface tension keeps short-wavelength waves moving quickly.
7. *Shallow-water gravity waves are non-dispersive.* Gravity waves on shallow water (which includes tidal waves on oceans!) travel at speed $\sqrt{gh}$, independent of wavelength.
8. *Froude number.* The Froude number, a ratio of kinetic to potential energy, determines the maximum speed of speedboats and of walking.

Problem 10.2 AM–GM

Prove the arithmetic mean–geometric mean inequality by another method than the circle in the text. Use AM–GM for the following problem normally done with calculus. You start with a unit square, cut equal squares from each corner, then fold the flaps upwards to make a half-open box. How large should the squares be in order to maximize its volume?



Problem 10.3 Impossible

How can tidal waves on the ocean (typical depth ~ 4 km) be considered shallow water?

Problem 10.4 Oven dish

Partly fill a rectangular glass oven dish with water and play with the waves. Give the dish a slight slap and watch the wave go back and forth. How does the wave speed time vary with depth of water? Does your data agree with the theory in this chapter?

Problem 10.5 Minimum-wave speed

Take a toothpick and move it through a pan of water. By experiment, find the speed at which no waves are generated. How well does it agree with the theory in this chapter?

Problem 10.6 Kelvin wedge

Show that the opening angle in a ship wake is $2 \sin^{-1}(1/3)$.

Problem 10.7 Semitones

Estimate 1.5^{14} using semitones and compare with the exact value.

Problem 10.8 Blackbody temperature of the earth

The earth's surface temperature is mostly due to solar radiation.

The solar flux $S \approx 1350 \text{ W m}^{-2}$ is the amount of solar energy reaching the top of the earth's atmosphere. But that energy is spread over the surface of a sphere, so $S/4$ is the relevant flux for calculating the surface temperature. Some of that energy is reflected back to space by clouds or ocean before it can heat the ground, so the heating flux is slightly lower than $S/4$. A useful estimate is $S' \sim 250 \text{ W m}^{-2}$.

Look up the Stefan–Boltzmann law (or see [Problem 6.9](#)) and use it to find the blackbody temperature of the earth.

Your value should be close to room temperature but enough colder to make you wonder about the discrepancy. Why is the actual average surface temperature warmer than the value calculated in this problem?

Problem 10.9 Xylophone notes

If you double the width, thickness, and length of a xylophone slat, what do you do to the frequency of the note that it makes?

Part 4

Backmatter



Long-lasting learning

The theme of this book is how to understand new fields, whether the field is known generally but is new to you; or the field is new to everyone. In either case, certain ways of thinking promote understanding and long-term learning. This afterword illustrates these ways by using an example that has appeared twice in the book – the volume of a pyramid.

Remember nothing!

The volume is proportional to the height, because of the drilling-core argument. So $V \propto h$. But a dimensionally correct expression for the volume needs two additional lengths. They can come only from b^2 . So

$$V \sim bh^2.$$

But what is the constant? It turns out to be $1/3$.

Connect to other problems

Is that 3 in the denominator new information to remember? No! That piece of information also connects to other problems.

First, you can derive it by using special cases, which is the subject of [Section 7.1](#).

Second, 3 is also the dimensionality of space. That fact is not a coincidence. Consider the simpler but analogous problem of the area of a triangle. Its area is

$$A = \frac{1}{2}bh.$$

The area has a similar form as the volume of the pyramid: A constant times a factor related to the base times the height. In two dimensions the constant is $1/2$. So the $1/3$ is likely to arise from the dimensionality of space.

That analysis makes the 3 easy to remember and thereby the whole formula for the volume.

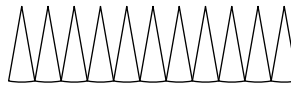
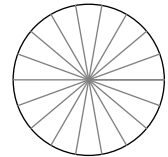
But there are two follow-up questions. The first is: Why does the dimensionality of space matter? The special-cases argument explains it because you need pyramids for each direction of space (I say no more for the moment until we do the special-cases argument in lecture!).

The second follow-up question is: Does the 3 occur in other problems and for the same reason? A related place is the volume of a sphere

$$V = \frac{4}{3}\pi r^3.$$

The ancient Greeks showed that the 3 in the $4/3$ is the same 3 as in the pyramid volume. To explain their picture, I'll use method to find the area of a circle then use it to find the volume of a sphere.

Divide a circle into many pie wedges. To find its area, cut somewhere on the circumference and unroll it into this shape:



Each pie wedge is almost a triangle, so its area is $bh/2$, where the height h is approximately r . The sum of all the bases is the circumference $2\pi r$, so $A = 2\pi r \times r/2 = \pi r^2$.

Now do the same procedure with a sphere: Divide it into small pieces that are almost pyramids, then unfold it. The unfolded sphere has a base area of $4\pi r^2$, which is the surface area of the sphere. So the volume of all the mini pyramids is

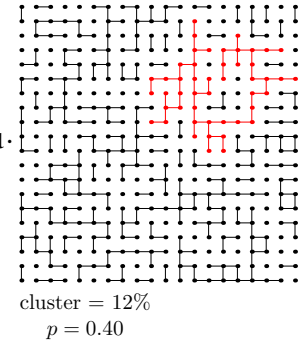
$$V = \frac{1}{3} \times \underbrace{\text{height}}_r \times \underbrace{\text{basearea}}_{4\pi r^2} = \frac{4}{3}\pi r^3.$$

Voilà! So, if you remember the volume of a sphere – and most of us have had it etched into our minds during our schooling – then you know that the volume of a pyramid contains a factor of 3 in the denominator.

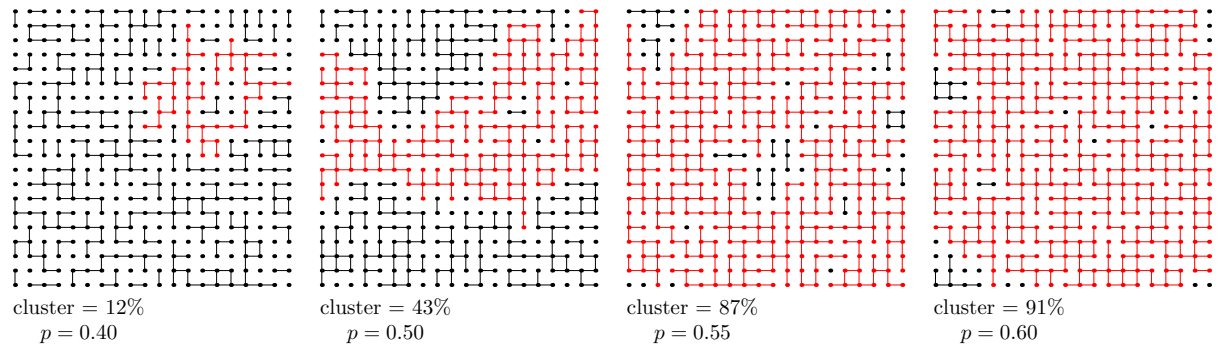
Percolation model

The moral of the preceding examples is to build connections. A physical illustration of this process is *percolation*. Imagine how oil diffuses through rock. The rock has pores through which oil moves from zone to zone. However, many pores are blocked by mineral deposits. How does the oil percolate through that kind of rock?

That question has led to an extensive mathematics research on the following idealized model. Imagine an infinite two-dimensional lattice. Now add bonds between neighbors (horizontal or vertical, not diagonal) with probability p_{bond} . The figure shows an example of a finite subsection of a percolation lattice where $p_{\text{bond}} = 0.4$. Its largest cluster – the largest set of points connected to each other – is marked in red, and contains 13 of the points.

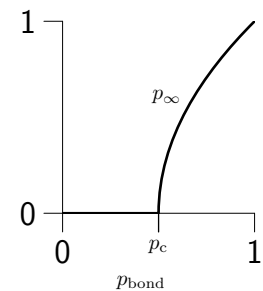


Here is what happens as p_{bond} increases from 0.40 to 0.50 to 0.55 to 0.60:



The largest cluster occupies more and more of the lattice.

For an infinite lattice, a similar question is: What is the probability p_{∞} of finding an infinite connected sublattice? That probability is zero until p_{bond} reaches a critical probability p_c . The critical probability depends on the topology (what kind of lattice and how many dimensions) – for the two-dimensional square lattice, $p_c = 1/2$ – but its existence is independent of topology. When $p_{\text{bond}} > p_c$, the probability of finding an infinite lattice becomes nonzero and eventually reaches 1.0.



An analogy to learning is that each lattice point (each dot) is a fact or formula, and each bond links two facts. For long-lasting learning, you want the facts to support each other via their connections. Let's say that you

want the facts to become part of an infinite and therefore self-supporting lattice. However, if your textbooks or way of learning means that you just add more dots – learn just more facts – then you decrease p_{bond} , so you decrease the chance of an infinite clusters. If the analogy is more exact than I think it is, you might even eliminate infinite clusters altogether.

The opposite approach is to ensure that, with each fact, you create links to facts that you already know. In the percolation model, you add bonds between the dots in order to increase p_{bond} . A famous English writer gave the same advice about life that I am giving about learning:

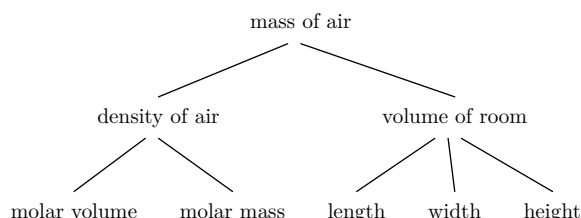
Only connect! That was the whole of her sermon. . . Live in fragments no longer! [E. M. Forster, *Howard's End*]

The ways of reasoning presented in this book offer some ways to build those connections. Bon voyage as you learn and discover new ideas and the links between them!

Solutions

Solution 1

One way to estimate the mass is to subdivide into the volume of the room and the density of air. The volume of the room subdivides into its length, height, and width. I remember that the density of air is roughly $1 \text{ g } \ell^{-1}$ because the value is needed often in estimation problems. Alternatively, you can use a useful fact from chemistry, that one mole of an ideal gas at standard temperature and pressure occupies 22 liters, and combine that fact with the molar mass of air. Using that method, the tree is



Now put values at the leaves.

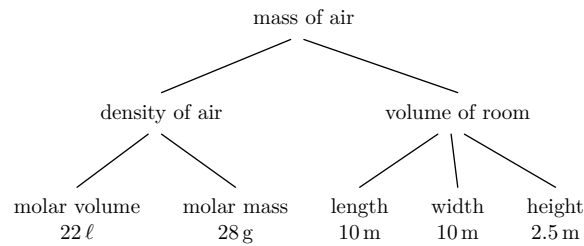
For the room dimensions, the MIT schedules office webpage gives the room area, but let's estimate the dimensions by eye. Most rooms are 8 or 9 feet high, or about 2.5 m. The room has about 10 rows, spaced around 1 m apart. So the length is about 10 m. The room is roughly square in aspect ratio, so the width is around 10 m as well. [The MIT page says that the area is 768 square feet, or about 77 m^2 . Our estimate of 100 m^2 is reasonably accurate.]

The molar volume for air (like any ideal gas) is 22 liters. The molar mass is, roughly, the molar mass of nitrogen, so about 14 g. But wait, nitrogen is diatomic, so the molar mass is 28 g.

The tree with values is:

Solutions

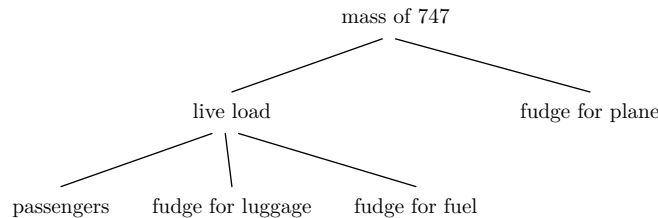
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Now propagate values upward. The volume of the room is 250 m^3 . The density of air is roughly $28/22 \text{ g l}^{-1}$, or roughly 1 kg m^{-3} . So the mass of air in the room is roughly 250 kg or about 500 pounds.

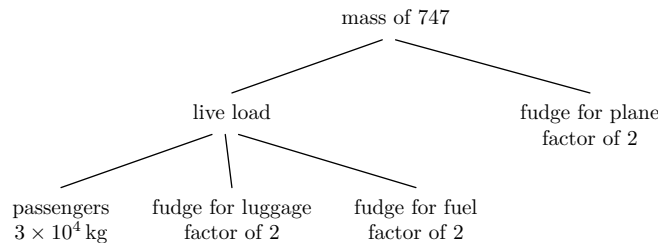
Solution 2

One method – and as usual there are many methods – is to estimate the mass of the passengers and then fudge that value to include the rest of the load (baggage and fuel), and then fudge that value to include the mass of the empty plane. Here is a tree to represent the method:



Now let's put values at the leaves. The plane holds about 400 passengers each weighing 70 kg , so $3 \times 10^4 \text{ kg}$. The fuel may be a factor of 2 or 3 larger than the passengers with luggage, although I am not confident about this number. And the plane itself, when empty, might weigh as much as all the load, so include another factor of 2 for the plane itself.

The tree with leaf values is



Now propagate values upward. The result is

$$m \sim 3 \times 10^4 \text{ kg} \times 2 \times 2 \times 2 \sim 2.5 \times 10^5 \text{ kg}.$$

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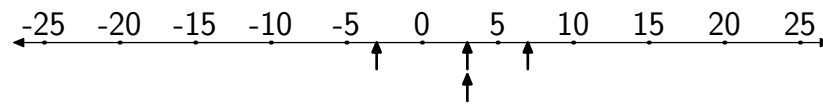
The actual maximum takeoff weight is (from Wikipedia) 4×10^5 kg. Not too bad (and more accurate than an initial guess without subdividing would have been).

Solution 3

When I flipped a coin 25 times (using a Python program), I ended 3 steps to the right from getting 14 heads and 11 tails.

Solution 4

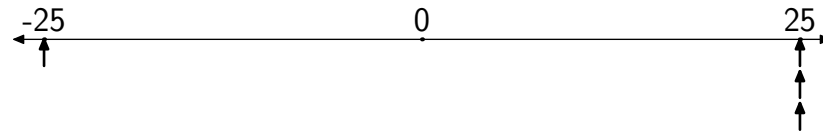
In the three repetitions I ended at 7, 3, and -3 . The four ending positions are

**Solution 5**

I ended at +25.

Solution 6

On the next repetitions, I ended at +25, -25 , and +25. Here are the four ending locations:

**Solution 7**

The marks in the first experiment are clustered much closer to the origin than they are in the second experiment.

The first experiment is analogous to dividing the estimate into 25 equal parts. The errors in the estimate of each part are not likely to point in the same direction. And that's why, when we did the experiment three more times, the results ended up near the origin.

The second experiment is analogous to making the estimate in one gulp, and not subdividing. And that's why the expected error is so large. The moral is to subdivide!

Solution 8

By having good aim or getting lucky, I won two tiny goldfish at our elementary school's annual fair. They lived happily for many years in our fish

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tank, eventually growing to 7 inches in length. The tank was perhaps 1 foot wide, 3 feet long, and 1.5 feet high, which is a volume of

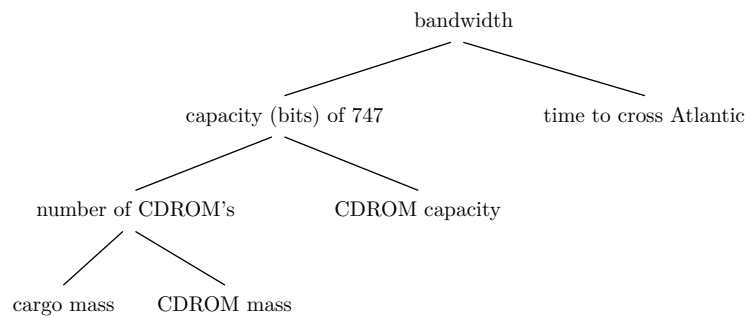
$$V \sim 0.3 \text{ m} \times 1 \text{ m} \times 0.5 \text{ m} \sim 0.15 \text{ m}^3.$$

If it filled two-thirds with water, that's 0.1 m^3 of water or 100 kg! The glass tank itself has a much lower mass so the main contribution is from the water.

I estimated the mass to only one digit, neglecting the mass of the empty tank. That's no worse than the errors from the length estimates. The tank itself is an old memory (from 30 years ago) and there's no need to overengineer the estimate.

Solution 9

Divide and conquer! Here's a tree on which to fill values:



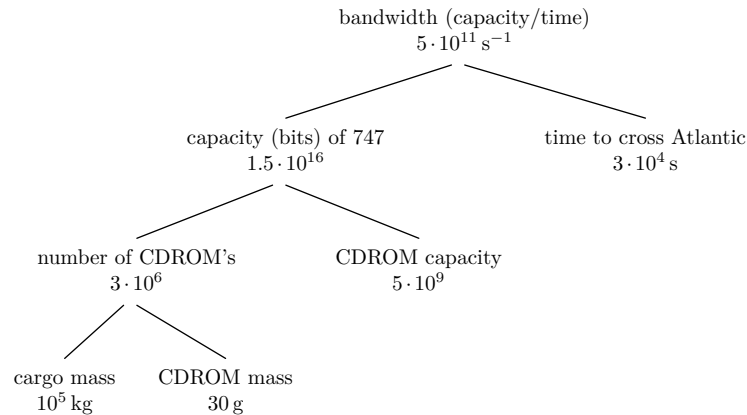
First I estimate the cargo mass. A 747 can easily carry about 400 people, each person having a mass (with luggage) of, say 140 kg. The total mass is

$$m \sim 400 \times 140 \text{ kg} \sim 6 \cdot 10^4 \text{ kg}.$$

A special cargo plane, with no seats or other frills for passengers, probably can carry 10^5 kg .

Here are the other estimates. A CDROM's mass is perhaps one ounce or 30 g. So the number of CDROM's is $3 \cdot 10^6$. The capacity of a CDROM is 600 MB or about $5 \cdot 10^9$ bits. The time to cross the Atlantic is about 8 hours or $3 \cdot 10^4 \text{ s}$.

Now propagate the values toward the root of the tree:



The bandwidth is 0.5 terabits per second.

Despite the large bandwidth offered by a 747 carrying CDROM's (not to mention DVDROM's), trans-Atlantic Internet connections go via undersea fiber-optic cables. Low latency is important!

Solution 10

Painting requires paint and labor. Let's estimate the ratio of labor to paint costs. A painter charges say \$35/hour. A gallon of paint costs maybe \$25. How much wall or ceiling area gets painted with that much paint? One method is to guess that paint goes onto walls as thickly as a sheet of paper (0.01 cm). So a gallon, which is roughly 4ℓ, can cover

$$\frac{4000 \text{ cm}^3}{0.01 \text{ cm}} \sim 4 \cdot 10^5 \text{ cm}^2 \sim 40 \text{ m}^2.$$

Since walls are roughly 4 m high, that's 10 m of wall, or one wall of a small classroom (or a medium-sized room in a house). It will take probably a couple hours to paint the wall, costing about \$70, so labor is more important than materials in the total cost.

The estimate in the last paragraph is 20 m² per hour, or about \$2/ m². Now estimate the classroom area. For a typical classroom, the wall and ceiling area is perhaps five times the area of the wall that separates the classroom from the corridor. So I'll estimate the wall area of the corridors. Walking the Infinite Corridor takes me about 3–4 minutes. Typical walking is 3 miles per hour, which is 1.5 m s⁻¹, makes its length 300 m. There are four storeys, maybe five if you count the basement. So that makes 1500 m of corridor. Multiply by a factor of 2 since there are walls on both sides; by a factor of 2 to account for the other parts of the main building; and by another factor of 2 to account for the other classroom buildings. That gives 10⁴ m of wall,

or about $4 \cdot 10^4 \text{ m}^2$ of wall. Multiplying by the factor of 5 gives $2 \cdot 10^5 \text{ m}^2$ of surface to be painted.

To paint $2 \cdot 10^5 \text{ m}^2$ would cost about $\$4 \cdot 10^5$.

Solution 11

The `ls -t` lists the files and subdirectories in a directory ordered by modification time with the most recently modified files at the beginning. The `head` selects the first ten lines, which means the first ten names. The `tac` reverses the order of the lines, so the 10th-most-recently-modified file (or subdirectory) comes first, then the 9th-most-recently-modified file, etc. with the most-recently-modified file at the end of the list.

Solution 12

Divide and conquer! The first step is to get rid of all the non-alphabetic characters and turn them into newlines. Then get rid of the empty lines, which occur either from empty lines in the original text or when consecutive non-alphabetic characters get turned into newlines. Then we'll have the words from the file, one word per line. This piece of the pipeline is

```
tr -cs 'a-zA-Z' '\n'
```

The `-c` option says that the list of characters is to be inverted (complemented). So `tr` will translate all characters except for the upper- and lowercase alphabetic characters `a-z` and `A-Z`. The backslash-`n` is the UNIX syntax for the newline character. The `-s` option tells `tr` to squeeze repeated translated characters into one copy of that character; therefore repeated newlines get turned into one newline, which gets rid of the empty lines.

To count the words, sort them and run `uniq`. `uniq` looks only at adjacent lines, which is why the word list needs to be sorted. In the simplest invocation, `uniq` print the first line from a series of duplicate lines. For example, feeding this input to `uniq`

```
the
the
the
how
the
how
how
```

produces

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Solutions

```
the
how
the
how
```

Giving `uniq` the `-c` switch tells it instead to count the duplicates. The same input to `uniq -c` produces

```
3 the
1 how
1 the
2 how
```

The pipeline so far is

```
tr -cs 'a-zA-Z' '\n' | sort | uniq -c
```

I want the top ten words, so I reverse sort the list numerically (so that the largest count is at the top) with `sort -nr`, then select the top ten lines with `head`.

The full pipeline is

```
tr -cs 'a-zA-Z' '\n' | sort | uniq -c | sort -nr | head
```

As a test, here is the result of applying that pipeline to an old email message about misconceptions about gravity on the moon. The full command is:

```
tr -cs 'a-zA-Z' '\n' < email.txt | sort | uniq -c | sort -nr | head
```

It produces this word-frequency list:

```
149 the
87 it
65 is
53 to
52 Moon
50 of
44 will
43 float
33 on
33 away
```

Solution 13

The integrand $x^3 e^{-x^2}$ is antisymmetric: Replacing x by $-x$ changes the function's sign. Therefore integrating it over a symmetric range such as -10 to 10 produces zero.

Solution 14

This integrand is also antisymmetric, so integrating it over a symmetric range such as $-\infty$ to ∞ produces zero.

[As a physics undergraduate, I spent many hours with the table of integrals that we knew affectionately as Gradshteyn. The table was so massive and complete that when we could not locate an integral in it, we suspected that the integral should be zero and went looking for the symmetry.]

Solution 15

Reversing the order of the terms is the symmetry operation because the sum is the same in reverse:

$$200 + 201 + 202 + \cdots + 300 = 300 + 299 + 298 + \cdots + 200.$$

Add these sums as follows:

$$\begin{array}{r} 200 + 201 + 202 + \cdots + 300 \\ + 300 + 299 + 298 + \cdots + 200 \\ \hline = 500 + 500 + 500 + \cdots + 500. \end{array}$$

There are 101 copies of the 500, so this duplicated sum is $500 \times 101 = 50500$. The original sum is one-half of the duplicated sum, so it is 25250.

A quick confirmation is the following Unix pipeline:

```
seq 200 300 | awk 'BEGIN {total=0}; {total += $1}; END {print total};'
```

which produces 25250.

Solution 16

This simulation in Python was written to be clear though not necessarily efficient. It uses a lattice to approximate the continuous sheet, and implements so-called relaxation: At each step, the temperature at each point is replaced by the average temperature of the neighbors. The main complications are:

1. The edges of the pentagon are held at fixed temperatures (10 degrees for four edges and 80 degrees for the fifth edge). However, the relaxation

step does not maintain those fixed values. So they are re-imposed after each sweep through the lattice.

2. Only one of the edges lies along a coordinate direction. The other four edges have funny slopes, and need to be rasterized. It is the identical problem to rendering lines on a laser printer: Which pixels get the toner? Bresenham's algorithm does the rasterization.

```
# Relaxation simulation of the temperature at the center of
the pentagon.
# Four edges are held at 10 degrees, and the fifth at 80 de-
grees.

from scipy import *

# rounds to nearest integer, and returns an integer
def intround(f):
    return int(round(f))

# Bresenham's algorithm: returns list of lattice points on the
line connecting
# r1 and r2
def line(r1, r2):
    x0, y0 = intround(r1[0]), intround(r1[1])
    x1, y1 = intround(r2[0]), intround(r2[1])
    points = []
    steep = abs(y1 - y0) > abs(x1 - x0)
    if steep:
        x0, y0 = y0, x0
        x1, y1 = y1, x1
    if x0 > x1:
        x0, x1 = x1, x0
        y0, y1 = y1, y0
    deltax = x1 - x0
    deltay = abs(y1 - y0)
    error = 0
    deltaerr = float(deltay) / deltax
    y = y0
    if y0 < y1:
        ystep = 1
    else:
        ystep = -1
```

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```

    for x in range(x0,x1+1):
        if steep:
            points.append((y,x))
        else:
            points.append((x,y))
        error += deltaerr
        if error >= 0.5:
            y += ystep
            error -= 1.0
    return points

def complex2pair(c):
    return (real(c),imag(c))

def set_edge_temps(grid):
    for e in lo_temp_edges:
        grid[e[0]][e[1]] = 10
    for e in hi_temp_edges:
        if e in lo_temp_edges:
            grid[e[0]][e[1]] = 45 # corner joining high- and
lo-temp edges
        else:
            grid[e[0]][e[1]] = 80

angle    = 72.0/180*pi          # 72 degrees
# use complex plane to find the vertices
r        = exp(angle*1j)
# pentagon vertices in the complex plane, with first vertex
duplicated
# at the end of the list
corners  = array([r**(i+0.25) for i in range(6)])
# translate pentagon into first quadrant
corners -= complex(min(real(corners)), min(imag(corners)))
corners *= 50                  # grid spacing (bigger means
finer spacing)
center   = sum(corners[0:5])/5 # center of pentagon

# use Bresenham's algorithm to find the lattice points on the
edges
lo_temp_edges = []
hi_temp_edges = []

```

```

for i in range(4):
    lo_temp_edges += line(complex2pair(corners[i]), complex2pair(corners[i+1]))
hi_temp_edges = line(complex2pair(corners[4]), complex2pair(corners[5]))

# figure out the grid dimensions
max_x = max([r[0] for r in lo_temp_edges+hi_temp_edges])
max_y = max([r[1] for r in lo_temp_edges+hi_temp_edges])
grid = zeros((max_x+1,max_y+1))

dirs = [(-1,0), (1,0), (0,1), (0,-1)]
while True:
    newgrid = zeros((max_x+1,max_y+1))
    set_edge_temps(grid)          # impose constraint
    for x in range(max_x+1):      # relax each location to avg
of its neighbors
        for y in range(max_y+1):
            total = n = 0
            for d in dirs:         # use each neighbor that's with-
in the grid
                try:
                    total += grid[x+d[0]][y+d[1]]
                    n += 1
                except: pass       # that neighbor was not in-
side the grid
            newgrid[x][y] = total/n # but save new value in
a new grid
    grid, newgrid = newgrid, grid  # swap new and old grid
    # print temperature at the center of the pentagon
    print grid[intround(real(center))][intround(imag(center))]

```

Solution 17

The original expression is antisymmetric in a and b : The result changes sign if you swap a and b .

The expansion has third-degree terms such as a^3 or a^2b . One category of third-degree terms is like a^3 and includes a^3 and b^3 . The antisymmetric combination is $a^3 - b^3$. The other category of third-degree terms is like a^2b and includes a^2b and ab^2 . The antisymmetric combination is $a^2b - ab^2$.

The expansion therefore has the antisymmetric form

$$(a - b)^3 = A(a^3 - b^3) + B(a^2b - ab^2)$$

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where A and B are constants to be determined.

Setting $b = 0$ shows that $A = 1$, because $(a - 0)^3 = A(a^3 - 0) + B(0 - 0)$ or $a^3 = Aa^3$.

To find B , think about the naive expansion of $(a - b)^3$. The basic expression $a - b$ has two terms, so $(a - b)^3$ has eight terms (before collecting like terms). So the absolute values of the coefficients of each term in the form

$$A(a^3 - b^3) + B(a^2b - ab^2)$$

have to add to eight. With $A = 1$, this requirement shows that $B = \pm 3$. The choice $B = -3$ gives the correct sign for the a^2b term (which has one negative factor from the $-b$).

So

$$(a - b)^3 = (a^3 - b^3) - 3(a^2b - ab^2).$$

Solution 18

The magnitude of the numerator is ω (assuming positive frequency). The magnitude of the denominator is $\sqrt{|\text{real part}|^2 + |\text{imaginary part}|^2}$ so it is $\sqrt{(1 - \omega^2)^2 + \omega^2/Q^2}$.

The ratio of magnitudes is $|G(\omega)|$:

$$|G(\omega)| = \frac{\omega}{\sqrt{(1 - \omega^2)^2 + \omega^2/Q^2}}.$$

Solution 19

When maximizing a parabolic function such as $y = x(6 - x)$, the symmetry is reflection about the line $x = 3$. In symbols, the transformation is $x_{\text{new}} = 6 - x$.

Let's transfer a few lessons from the parabola example to the problem of maximizing the gain. In the parabola example, the symmetry is a reflection about an interesting point (there, the point halfway between the two roots $x = 0$ and $x = 6$). Analogously, an interesting frequency is $\omega = 1$ because it makes the real part of the denominator in $G(\omega)$ go to zero, and making the real part go to zero helps minimize the denominator.

Therefore reflecting about $\omega = 1$ is worth trying, perhaps $\omega_{\text{new}} = 1 - \omega$. For frequencies, however, differences are not as important as ratios. For example, a musical octave is a factor of 2 in frequency, rather than a difference. So reflect in a multiplicative way: $\omega_{\text{new}} = \omega^{-1}$.

This transformation works either in $G(\omega)$ or in the magnitude $|G(\omega)|$. It's slightly easier in $G(\omega)$:

$$G(\omega) = \frac{j\omega}{1 + j\omega/Q - \omega^2} \mapsto H(\omega_{\text{new}}) = \frac{j/\omega_{\text{new}}}{1 + j/Q\omega_{\text{new}} - 1/\omega_{\text{new}}^2}.$$

Multiply numerator and denominator by ω_{new}^2 :

$$H(\omega_{\text{new}}) = \frac{j\omega_{\text{new}}}{\omega_{\text{new}}^2 + j\omega_{\text{new}}/Q - 1},$$

which is the same function as $G(\omega)$, except for negating the real part in the denominator. Negating the real part in the denominator doesn't affect the magnitude of the denominator, so $|H(\omega_{\text{new}})|$ has the same form as $|G(\omega)|$.

Solution 20

Since $\omega_{\text{new}} = 1/\omega$, the maximum value of ω_{new} will be ω_{max}^{-1} . That's one equation.

Since the two magnitudes $|G(\omega)|$ and $|H(\omega_{\text{new}})|$ are the same function, the maximum value of ω_{new} is also the maximum value of ω . That's the second equation.

Together they produce $\omega = \omega_{\text{new}} = 1$ (ignoring the negative-frequency solution $\omega = -1$). At that frequency, $|G(\omega)|$ is Q . For the electrical and mechanical engineers: The quality factor Q is also the gain at resonance.

Solution 21

A direct differentiation of $|G(\omega)|$ is too awful for words, and I cannot make myself do it. Instead I'll massage the expression until differentiating is not horrible or maybe not even needed.

First, put the ω from the numerator into the denominator by multiplying numerator and denominator by $1/\omega$:

$$|G(\omega)| = \frac{\omega}{\sqrt{(1 - \omega^2)^2 + \omega^2/Q^2}} = \frac{1}{\sqrt{(\omega^{-1} - \omega)^2 + 1/Q^2}}.$$

Second, find the extremum of an equivalent, simpler expression. Maximizing $1/\sqrt{f(\omega)}$ is equivalent to maximizing $1/f(\omega)$. And maximizing $1/f(\omega)$ is equivalent to minimizing $f(\omega)$. So I'll find the extremum of

$$(\omega^{-1} - \omega)^2 + 1/Q^2.$$

Furthermore, the $1/Q^2$ doesn't affect the location of the extremum, so instead I minimize $(\omega^{-1} - \omega)^2$. Even better, the squaring does not affect the

location of the extremum, so I minimize the absolute value of $\omega^{-1} - \omega$. Its absolute value can never fall below zero, and it equals zero when $\omega = 1$. So $\omega = 1$ is the location of the maximum of $|G(\omega)|$. No need for differentiation!

Solution 22

Whatever coordinate change I make, I will leave the x axis alone because the I_{xx} component is already separated from the y - and z submatrix. That submatrix is

$$\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$$

I have to figure out how changing the coordinate system changes this submatrix. Rather than find the coordinate change explicitly, I use invariants to avoid that computation.

One invariant of any matrix, not just of this 2×2 matrix, is its determinant. Another invariant is its trace (the sum of the diagonal elements). In the nasty coordinate system, the trace of the y - and z submatrix is $5 + 5 = 10$. So the trace is 10 in the nice coordinate system. The determinant is $5 \times 5 - 4 \times 4 = 9$, so the determinant is 9 in the nice coordinate system.

Those facts are sufficient to deduce the submatrix in the nice coordinate system (without needing to figure out what the nice coordinate system is). In the nice coordinate system, the 2×2 submatrix looks like

$$\begin{pmatrix} I_{yy} & 0 \\ 0 & I_{zz} \end{pmatrix}$$

So I need to find I_{yy} and I_{zz} such that

$$I_{yy} + I_{zz} = 10 \quad (\text{from the trace invariant})$$

and

$$I_{yy} I_{zz} = 9 \quad (\text{from the determinant invariant})$$

The solution is $I_{yy} = 1$ and $I_{zz} = 9$ (or vice versa). So the inertia tensor becomes

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

Solution 23

The object has three principal axes, each with a different moment of inertia. If the object is rectangular and uniform density, the three axes must have different lengths. Most books fit into this category. They have a short axis that passes perpendicularly through the pages (this axis is the one with the highest moment of inertia). The medium-length axis is perpendicular to the spine. And the long axis is parallel to the spine.

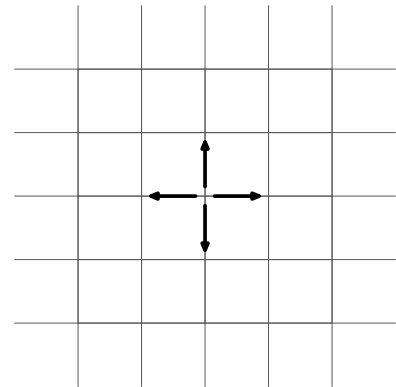
Solution 24

I'd like to find the current flowing through the resistor when 1 A is sent into one terminal of the ohmmeter and removed from its other terminal. The solution has two steps, each subtle:

1. Break the resistance-measuring experiment into two parts, each having a lot of symmetry.
2. Analyze those parts using symmetry.

The current distribution that results from the full resistance-measuring experiment is not sufficiently symmetric because it has a preferred direction along the selected resistor. However, if I break the experiment into two parts – inserting current and removing current – then each part produces a symmetric current distribution.

By symmetry – because all four coordinate directions are equivalent – inserting 1 A produces $1/4$ A flowing in each coordinate direction away from the terminal. Let's call this terminal the positive terminal. So inserting the 1 A at the positive terminal produces $1/4$ A through the selected resistor, and this current flows away from the positive terminal.

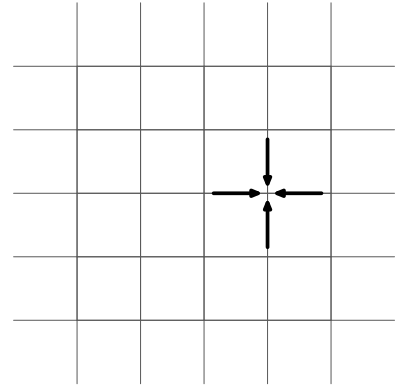


Solutions

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By symmetry, removing 1 A produces 1/4 A in each coordinate direction, flowing toward the terminal. Let's call this terminal the negative terminal. So removing 1 A produces 1/4 A through the selected resistor, flowing toward the negative terminal. Equivalently, it produces 1/4 A flowing away from the positive terminal.

Now superimpose the two pictures to reproduce the experiment of measuring the resistance. The experiment produces 1/2 A through the resistor, flowing from the positive to the negative terminal. The voltage across the resistor is the current times its resistance, so the voltage is 1/2 V. Since a 1 A test current produces a 1/2 V drop, the effective resistance is 1/2 Ω .



If you want an even more difficult problem: Find the resistance measured across a diagonal!

Solution 25

The weight of the raindrop is the density times the volume times g :

$$W \sim \rho r^3 g,$$

where I neglect dimensionless factors such as $4\pi/3$.

At terminal velocity, the weight equals the drag. The drag is

$$F \sim \rho_{\text{air}} v^2 A \sim \rho_{\text{air}} v^2 r^2.$$

Equating the weight to the drag gives an equation for v and r :

$$\rho_{\text{air}} v^2 r^2 \sim \rho r^3 g,$$

so $v \propto r^{1/2}$.

Bigger raindrops fall faster but – because of the square root – not much faster.

Solution 26

With the g and the densities, the terminal velocity is

$$v \sim \sqrt{\frac{\rho}{\rho_{\text{air}}}} gr.$$

A typical raindrop has a diameter of maybe 6 mm, so $r \sim 3$ mm. Since the density ratio between water and air is roughly 1000,

$$v \sim \sqrt{1000 \times 10 \text{ m s}^{-2} \times 3 \cdot 10^{-3} \text{ m}} \sim 5 \text{ m s}^{-1}.$$

Solution 27

First convert the speed into a more familiar value: 11 mph (miles per hour). If one drives at a speed v_{car} , then raindrops appear to move at an angle $\arctan(v_{\text{car}}/v)$. When $v_{\text{car}} = v$, the drops come at a 45-degree angle. So one way to measure the terminal speed is to drive in a rainstorm, slowly accelerating while the passenger says when the drops come at a 45-degree angle.

You could also run in a rainstorm and note the speed at which a small umbrella has to be at 45 degrees to keep you perfectly dry.

Solution 28

The heights are:

- Mars: 27 km (Mount Olympus)
- earth: 9 km (Mount Everest)
- Venus: 11 km (Maxwell Montes)

One pattern is that the large planets (earth and Venus) have short mountains, at least short compared to Mount Olympus at a huge height of 27 km.

Large planets presumably have stronger gravitational fields at their surface, which keeps the mountains closer to the ground. The derivation in lecture on mountain heights dropped the dependence on g because we looked only at mountains on earth. Here's the same derivation but retaining g . The weight of a mountain of size l is $W \propto gl^3$, so the pressure at the base is $p \propto gl^3/l^2 \sim gl$. When the pressure exceeds the maximum pressure that rock can support, the mountain can no longer grow upward. So the maximum height l depends inversely on g :

$$l \propto g^{-1}.$$

To test that analysis, here are the gravitational field strengths on the three planets:

- Mars: 3.7 m s^{-2}
- earth: 10 m s^{-2}
- Venus: 8.9 m s^{-2}

The product gl for each planet should be the same, and it roughly is:

Solutions

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- Mars: $10^5 \text{ m}^2 \text{ s}^{-2}$
- earth: $0.9 \cdot 10^5 \text{ m}^2 \text{ s}^{-2}$
- Venus: $0.98 \cdot 10^5 \text{ m}^2 \text{ s}^{-2}$

Fun question: Why aren't mountains on the moon 60 km tall (since the Moon's surface gravity is about one-sixth of earth's surface gravity)?

Solution 29

In the ratio $\rho v^2 A d / m_{\text{car}} v^2$, the v^2 divide out leaving $\rho A d / m_{\text{car}}$, where ρ is the air density. Since A is the cross-sectional area of the car, $A d$ is the volume of air that the car displaces, and $\rho A d$ is the mass of that air. So

$$\frac{E_{\text{drag}}}{E_{\text{kinetic}}} \sim \frac{\rho v^2 A d}{m_{\text{car}} v^2} = \frac{\rho A d}{m_{\text{car}}} = \frac{\text{mass of the air displaced}}{\text{mass of the car}}.$$

An alternative equivalence comes from writing the mass of the car as $\rho_{\text{car}} A l_{\text{car}}$. Making that substitution and dividing out the v^2 gives

$$\frac{\rho v^2 A d}{m_{\text{car}} v^2} = \frac{\rho_{\text{air}} A d}{\rho_{\text{car}} A l_{\text{car}}} = \frac{\rho_{\text{air}}}{\rho_{\text{car}}} \frac{d}{l_{\text{car}}}.$$

Solution 30

A typical car has mass $m_{\text{car}} \sim 10^3 \text{ kg}$, cross-sectional area $A \sim 2 \text{ m} \times 1.5 \text{ m} = 3 \text{ m}^2$, and length $l_{\text{car}} \sim 4 \text{ m}$. So

$$\rho_{\text{car}} \sim \frac{m_{\text{car}}}{A l_{\text{car}}} \sim \frac{10^3 \text{ kg}}{3 \text{ m}^2 \times 4 \text{ m}} \sim 10^2 \text{ kg m}^{-3}.$$

Since $\rho_{\text{car}} / \rho_{\text{air}} \sim 100$, the ratio

$$\frac{\rho_{\text{air}}}{\rho_{\text{car}}} \frac{d}{l_{\text{car}}}$$

becomes 1 when $d / l_{\text{car}} \sim 100$, so $d \sim 400 \text{ m}$.

This distance d is significantly farther than the distance between stop signs or stoplights on city streets. In Manhattan, for example, 20 east–west blocks are one mile, giving a spacing of approximately 80 m. So air resistance is not a significant loss in city driving. Instead the loss comes from engine friction, rolling resistance, and braking.

However, the distance d is comparable to the exit spacing on urban highways. So when you drive on the highway for even a few exit distances, air resistance is a significant loss.

Interestingly, highway fuel efficiencies are higher than city fuel efficiencies, even though drag gets worse at the higher, highway speeds, and presumably engine friction and rolling resistance also get worse at higher speeds. Only one loss mechanism, braking, is less prevalent in highway than in city driving. So braking must cause a significant loss in city driving. Regenerative braking, for hybrid or electric cars, should significantly improve fuel efficiency in city driving.

Solution 31

The force of gravity on an object of mass m is $F = mg$. By Newton's law of gravitation, it is also $F = GMm/R^2$, where M is the mass of the planet and G is Newton's constant of gravitation. Therefore the gravitational acceleration is $g = GM/R^2$. Since $M = \rho(4\pi/3)R^3$, the gravitational acceleration is

$$g = \frac{G\rho(4\pi/3)R^3}{R^2} \propto \rho R.$$

In the last step, G vanished and the equals sign got replaced by a proportionality, which is okay since G is the same for all planets in the universe.

Solution 32

Using the proportionality, the ratio of gravities is

$$\frac{g_{\text{moon}}}{g_{\text{earth}}} = \frac{\rho_{\text{moon}}}{\rho_{\text{earth}}} \frac{R_{\text{moon}}}{R_{\text{earth}}}.$$

The factors are ratios of densities or radii, so they are dimensionless.

Solution 33

I'll first assume that earth and moon rock are the same. So $\rho_{\text{moon}}/\rho_{\text{earth}} \sim 1$.

The earth's radius is worth memorizing once you've derived it. Here's one way to derive it. The distance from Boston to San Francisco is about 3000 miles, and the cities are separated by three time zones. So the sun 'travels' about 1000 miles per time zone (per hour). Since one day has 24 time zones, the sun's travel around the earth is about 24,000 miles. That value is the circumference $2\pi R_{\text{earth}}$, so $R_{\text{earth}} \sim 4 \cdot 10^3$ mi (since $\pi \sim 3$) or $6.4 \cdot 10^3$ km. This estimate neglects a trigonometric factor due Boston not being on the equator, but it makes other errors, and they cancel (surprise!): The true value of the mean radius is 6373 km.

The moon's radius needs a different analysis. I can just cover the moon with my index finger at arm's length. So the moon subtends an angle

Solutions

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$$\theta \sim \frac{\text{width of my finger}}{\text{my arm length}} \sim \frac{1 \text{ cm}}{1 \text{ m}} \sim 0.01.$$

So the *diameter* of the moon is roughly θd , where $d \sim 4 \cdot 10^8 \text{ m}$ is the distance to the moon, and the radius is therefore $R_{\text{moon}} \sim 2 \cdot 10^6 \text{ m}$. If the moon is hidden, you can (carefully!) use the sun instead because it has the same angular size as the moon – which is the explanation for total solar eclipses.

The density and radii factors produce

$$\frac{g_{\text{moon}}}{g_{\text{earth}}} = \underbrace{1}_{\text{densities}} \times \underbrace{\frac{1}{3}}_{\text{radii}} = \frac{1}{3}.$$

So $g_{\text{moon}} \sim 3 \text{ m s}^{-2}$.

Solution 34

The true value is $g_{\text{moon}} \sim 1.6 \text{ m s}^{-2}$. So the estimate is too high by a factor of 2. The radii estimates are fairly accurate, so the equal-density assumption must be significantly wrong. So the moon's density is much less than the earth's. The actual values are $\rho_{\text{moon}} \sim 3.4 \text{ g cm}^{-3}$ and $\rho_{\text{earth}} \sim 5.5 \text{ g cm}^{-3}$. However, ρ_{moon} is comparable to the density of rock in the earth's crust. Perhaps the moon was once part of the earth's crust, which is still a viable theory of the moon's origin.

Solution 35

A roundtrip ticket from New York to San Francisco costs roughly \$400. The journey is about 2500 miles each way, so a 5000-mile journey costs about \$500 (rounding up the \$400 to make the math easier). That's about 10 cents/mile. Perhaps one-half of that cost is fuel. [Although the service – in the air, on the phone, and at the counter – is so lousy due to understaffing that perhaps two-thirds of the cost being fuel would be a better estimate!] At 5 cents/mile for fuel, and at \$3/gallon for fuel, the fuel efficiency is 60 passenger-miles per gallon.

Solution 36

The 747 can hold about 400 people, so the fuel efficiency is

$$\frac{400 \text{ passengers} \times 1.3 \cdot 10^4 \text{ km}}{2 \cdot 10^5 \ell} \times \frac{1 \text{ mile}}{1.6 \text{ km}} \times \frac{4 \ell}{1 \text{ gallon}} \sim 65 \text{ passenger-miles per gallon.}$$

This estimate is amazingly close to the estimate from using the ticket price!

Solution 37

The fuel efficiency of a medium-sized car (holding one person, which is typical for California) is roughly 30 passenger-miles per gallon. So both

fuel-efficiency estimates in this problem give a fuel efficiency that is a factor of 2 higher than the result from lecture – not too bad considering how much we neglected (drag coefficient and lift being the main ones) when we estimated the efficiency.

Solution 38

According to the Buckingham Pi theorem, five quantities composed of three independent dimensions make two independent dimensionless groups.

The question did not ask you to choose the dimensionless groups. But it is useful to see what they could be. The following pair is a reasonable choice for the two groups:

$$\Pi_1 \equiv \frac{\hbar^2}{m_e a_0 (e^2/4\pi\epsilon_0)} \quad \text{and} \quad \Pi_2 \equiv \frac{e^2/4\pi\epsilon_0}{\hbar c}.$$

The groups are often called Π variables, following a tradition started by Buckingham, author of the Buckingham Pi theorem.

In the preceding choice of groups, the second group Π_2 is the fine-structure constant α .

Solution 39

Five quantities composed of three independent dimensions make two independent dimensionless groups. Here is a reasonable combination:

$$\Pi_1 \equiv \frac{kT^2}{m} \quad \text{and} \quad \Pi_2 \equiv \frac{kx_0}{mg}.$$

It turns out that the second group Π_2 does not affect the period T . However, dimensional analysis does not tell you that result; it has to be derived by physical thinking.

Solution 40

Three quantities composed of two dimensions (length and time) produce one independent dimensionless group. A reasonable choice is

$$\Pi_1 \equiv \frac{v^2}{gh}.$$

That ratio – except for a factor of 2 – has a physical interpretation as the ratio of kinetic energy on impact to the potential energy at the start.

Solution 41

These three quantities are composed of three dimensions (mass, length, and time), so there should be zero dimensionless groups! However, there is at least one group: W/mg is dimensionless.

What went wrong is that the three quantities are composed of two *independent* dimensions: mass M and acceleration LT^{-2} . So the Buckingham Pi theorem predicts one independent dimensionless group. A reasonable choice for it is W/mg .

Solution 42

An exponent, for example $-\beta x^2$, the question, 'How many times do I multiply the base by itself?' Here the base is e , but the principle applies to any exponent.

Pretend that x is a length. The dimensions of dx are the same as the dimensions of x , so dx is also a length. The exponential is dimensionless since it's the product, $-\beta x^2$ times, of the dimensionless number e . So the integral, which is the sum of little lengths (from the dx), is also a length.

Now determine which choice has dimensions of length. Since x is a length, $[\beta] = L^{-2}$ in order that βx^2 be dimensionless. So $\beta^{-1/2}$ is a length; any other power is something else.

The only reasonable choice is $\sqrt{\pi}\beta^{-1/2}$. I might have been unkind by giving you a choice with the wrong dimensionless constant alongside the correct power of β . But not this time: The factor of $\sqrt{\pi}$ is correct.

Solution 43

Looking at the Bohr radius

$$a_0 = \frac{\hbar^2}{m_e(e^2/4\pi\epsilon_0)},$$

I see two powers of $\hbar$ just asking to be joined with two powers of c to make $(\hbar c)^2$. To add two powers of c without changing the fraction, multiply the denominator by c^2 as well. The electron mass m_e is happy to join with two powers of c to make $m_e c^2$, the rest energy of the electron.

Therefore,

$$a_0 = \frac{\hbar^2 c^2}{m_e c^2 (e^2/4\pi\epsilon_0)},$$

Now use

$$\alpha = \frac{e^2/4\pi\epsilon_0}{\hbar c}$$

to simplify even more:

$$a_0 = \frac{\hbar c}{m_e c^2} \underbrace{\frac{\hbar c}{e^2/4\pi\epsilon_0}}_{\alpha^{-1}} = \alpha^{-1} \frac{\hbar c}{m_e c^2}.$$

The pieces of this form are easy to remember:

$$a_0 = \alpha^{-1} \frac{\hbar c}{m_e c^2} \sim 10^2 \times \frac{2000 \text{ eV}\text{\AA}}{0.5 \cdot 10^6 \text{ eV}} \sim 0.5\text{\AA}.$$

Solution 44

With Z protons pulling on one electron, the electrostatic energy is $Ze^2/4\pi\epsilon_0$. So instead of using $e^2/4\pi\epsilon_0$ as one of the quantities, the dimensional analysis should use $Ze^2/4\pi\epsilon_0$. The other quantities are unchanged. The Z propagates along with the e^2 through the calculation of the radius a_Z and the energy $E(Z)$.

Since the radius a_0 has one factor of e^2 in the denominator, the a_Z picks up a factor of Z in the denominator relative to a_0 :

$$a_Z = \frac{a_0}{Z}.$$

Since the energy E_0 has a factor of e^4 in the numerator, the energy $E(Z)$ picks up a factor of Z^2 :

$$E(Z) = E(1) \times Z^2.$$

Solution 45

To increase the energy of the electron by a factor of Z^2 , the speed must increase by a factor of Z . So the innermost electron moves with speed $Z\alpha c$.

Solution 46

When $Z\alpha c \sim c$, the electron becomes significantly relativistic and permits pair creation to destabilize the nucleus. So the maximum Z is roughly α^{-1} or about 137.

The heaviest stable nucleus is uranium with $Z = 92$, so the estimate is not too bad.

Solution 47

Zero! There are three quantities and three independent dimensions, so the Buckingham Pi theorem says there will be no dimensionless groups.

Solution 48

Radiation travels at the speed of light. Furthermore, radiation is a relativistic phenomenon. Relativity means that magnetic fields are electric fields

seen from a moving reference frame. And radiation is an alternation of electric and magnetic fields, like a cat chasing its tail, so radiation needs relativity. And relativity needs the speed of light.

Therefore, I should add c to the list of variables.

Solution 49

Four quantities composed of three dimensions produce one independent group. The dimensions of power P are ML^2T^{-3} . The dimensions of $q^2/4\pi\epsilon_0$ are energy times distance, which is ML^3T^{-2} . The dimensions of acceleration a are LT^{-2} . And the dimensions of c are LT^{-1} .

In order to cancel the mass dimensions, a dimensionless combination will contain $P/(q^2/4\pi\epsilon_0)$, which has dimensions of $L^{-1}T^{-1}$. The only way to make those dimensions from a and c is a^2/c^3 . So a dimensionless group is

$$\Pi_1 \equiv \frac{P/(q^2/4\pi\epsilon_0)}{a^2/c^3} = \frac{Pc^3}{a^2(q^2/4\pi\epsilon_0)}.$$

The radiated power P must therefore be

$$P \sim \frac{a^2 q^2 / 4\pi\epsilon_0}{c^3}.$$

This result explains why radiation (radio waves, light, etc.) is such a good way to transmit information for a long distance. To see how, I'll use the power to estimate the electric field at a distance r from the accelerating (and radiating) charge.

The power P is the same no matter how far away one is from the charge because all the radiated energy escapes to infinity. Pretend that the angular distribution of the power is isotropic. Then the radiated power *per area* at a distance r is

$$\mathcal{P} \sim \frac{P}{\text{area of sphere with radius } r} \sim \frac{P}{r^2}.$$

The power per area is also known as the energy flux, which is energy per area per time. Energy flux is the propagation speed times the energy per volume (the energy density) $\mathcal{E}$ in the electric field:

$$\mathcal{P} \propto c\mathcal{E}.$$

Hard to believe? Check the dimensions!

The energy density in the electric field is proportional to the square of the field E :

$$\mathcal{E} \propto E^2.$$

This relation is analogous to the relation that the energy in a spring is proportional to the square of the amplitude.

Put these relations together to find the relation between energy flux and electric field:

$$\mathcal{P} \propto cE^2.$$

Since $\mathcal{P} \sim P/r^2$:

$$E \propto r^{-1}.$$

Compare this distance dependence with the similar result for electrostatics: There the electric field is proportional to r^{-2} . So, by accelerating a charge, we make an electric field that falls off much more slowly than it would from just electrostatics. The importance of that change grows as r grows. For large r – from radio stations and, especially, from stars – radiation is the only way to get a message to us. What a difference that change in the exponent makes!

Solution 50

Four quantities composed of three independent dimensions make one independent dimensionless group. The only quantities whose dimensions contain mass are E and ρ . So E and ρ appear in the group as E/ρ , whose dimensions are L^5T^{-2} . Therefore the following choice is dimensionless:

$$\Pi_1 \equiv \frac{Et^2}{\rho R^5}.$$

Solution 51

With only one dimensionless group, the most general statement connecting those four quantities is

$$\frac{Et^2}{\rho R^5} \sim 1.$$

or

$$E \sim \frac{\rho R^5}{t^2}.$$

For each row of data in the table, I'll estimate $\rho R^5/t^2$, using $\rho \sim 1 \text{ kg m}^{-3}$:

Solutions

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t (ms)	R (m)	E (10 ¹³ J)
3.26	59.0	6.7
4.61	67.3	6.5
15.0	106.5	6.1
62.0	185.0	5.6

The data are not perfectly consistent about the predicted blast energy E , but they hover pretty closely around $6 \cdot 10^{13}$ J.

Solution 52

One kiloton of TNT is

$$1 \text{ kiloton} \times \frac{10^9 \text{ g}}{1 \text{ kiloton}} \times \frac{4 \cdot 10^3 \text{ J}}{1 \text{ g}} \sim 4 \cdot 10^{12} \text{ J}.$$

The predicted yield of $6 \cdot 10^{13}$ J is roughly 15 kiltons, in close agreement with the classified value of 20 kiltons.

If I'd used the more accurate density $\rho \sim 1.3 \text{ kg m}^{-3}$, then I'd have found $E \sim 19.5$ kiltons, a result that is far too accurate for the number of approximations contained in it!

Solution 53

I'll use the blast energy E , the radius R , and the air density ρ to estimate out how fast the fireball expands when it has radius R .

For a simple model, imagine that the fireball expands as fast as the speed of sound. Of course, the speed of sound is much higher in the fireball than in normal air, because the fireball has a huge pressure due to the blast.

The energy density $\mathcal{E}$ in the fireball is the blast energy E divided by the volume of the fireball, so $\mathcal{E} \sim E/R^3$. The energy density is the pressure, give or take a dimensionless constant. Sound speed depends on pressure and density (check the dimensions):

$$c_s \sim \sqrt{\frac{\text{pressure}}{\text{density}}}.$$

So the sound speed, which is the rate at which the fireball expands, is

$$c_s \sim \sqrt{\frac{E}{\rho R^3}}.$$

Next assume that the expansion speed is constant, even though it falls as fireball grows. Then the radius is $R \sim tc_s$. So

$$R \sim t \sqrt{\frac{E}{\rho R^3}},$$

or

$$E \sim \frac{\rho R^5}{t^2},$$

as derived using dimensions.

Solution 54

The most useful special cases here are $a \rightarrow 0$ and $a \rightarrow \infty$. When a is zero, the Gaussian becomes the flat line $y = 1$, which has infinite area. The first choice, $\sqrt{\pi a}$, goes to zero in this limit, so it cannot be right. The second choice, $\sqrt{\pi/a}$, has the correct behavior.

The limit $a \rightarrow \infty$ gives the same conclusion: The first choice cannot be right, and the second one might be right.

Solution 55

The easiest special case is $a \rightarrow \infty$. In that limit, the integrand is zero everywhere, so the integral is zero. The first and third choices are therefore incorrect.

To decide between the second and fourth choices, use the special case $a = 1$. The integral becomes

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$$

The integral is $\arctan x$. At ∞ it contributes $\pi/2$, and at $-\infty$ it subtracts $-\pi/2$, so the integral is π . Only the second choice, π/a , has the correct behavior.

[The problem statement had an error, which one of you found (thank you!): The problem should either have said $a \geq 0$ or have used $|a|$ in the candidate answers.]

Solution 56

Special cases are useful in debugging programs. The easiest cases are often $n = 0$ or $n = 1$. Let's try $n = 0$ first. In the first program, the $2n + 1$ in the loop condition means that $i = 1$ is the only case, so the total becomes 1. Whereas the sum of the first 0 odd numbers should be zero! So the first program looks suspicious.

Let's confirm that analysis using $n = 1$. The first program will have $i = 1$ and $i = 3$ in the loop, making the total $1 + 3 = 4$. The second program will have $i = 1$ in the loop, making the total 1. Since the correct answer is 1, the first program has a bug, and the second program looks sound.

Solution 57

From an earlier problem set, a raindrop falls at about 10 m s^{-1} and it has a radius of roughly 3 mm. So the Reynolds number is

$$\text{Re} = \frac{rv}{\nu} \sim \frac{3 \cdot 10^{-3} \text{ m} \times 10 \text{ m s}^{-1}}{1.5 \cdot 10^{-5} \text{ m}^2 \text{ s}^{-1}} \sim 2000.$$

Solution 58

Using reasonable guesses for the flight speed and size:

$$\text{Re} \sim \frac{10^{-3} \text{ m} \times 1 \text{ m s}^{-1}}{10^{-5} \text{ m}^2 \text{ s}^{-1}} \sim 100.$$

Even a mosquito experiences high-Reynolds-number drag. It's not quite turbulent flow, which happens around $\text{Re} \sim 10^3$, but it's still significantly higher than the value for low-Reynolds-number (Stokes) drag.

Solution 59

The drag coefficient is

$$c_d \equiv \frac{F}{\frac{1}{2}\rho v^2 A},$$

where $A = \pi r^2$ is the cross-sectional area. So

$$c_d = \frac{6\pi\rho\nu r}{\frac{1}{2}\rho v^2 \pi r^2} = 12 \frac{\nu}{vr} = \frac{12}{\text{Re}}.$$

Solution 60

Three useful special cases are $a = 0$, $b = 0$, and $a = b$. Each special case is useful because it simplifies the figure. When $a = 0$, the figure is the original square-based pyramid with base side b . When $b = 0$, the figure is an inverted square-based pyramid with base side a . When $a = b$, the figure is a rectangular prism.

The first candidate for the volume, $hb^2/3$, works when $a = 0$ but fails when $b = 0$ or $a = b$. The second candidate fails when $a = 0$ or $a = b$.

The third candidate works when $a = 0$ or $b = 0$. Great! However, it fails when $a = b$ because it predicts that the rectangular prism has volume

$2hb^2/3$ rather than hb^2 . In contrast, the fourth candidate works when $a = b$ but fails when $a = 0$ or $b = 0$.

So, we need a new formula. Making formulas just with a^2 and b^2 does not provide enough freedom to accommodate the three special cases. To think of another term, look at what a^2 and b^2 have in common. They are both quadratic. A third quadratic term, not yet used, is ab . So what about a formula like

$$V = \frac{1}{3}h(a^2 + \beta ab + b^2),$$

where β is an unknown constant. The $a = 0$ and $b = 0$ cases will work because of the coefficient of one-third. So now choose the coefficient of β to make the $a = b$ special case work. When $a = b$, the proposed volume becomes

$$V = \frac{2 + \beta}{3}hb^2.$$

Since the correct volume is hb^2 , the only possibility is $\beta = 1$. Therefore

$$V = \frac{1}{3}h(a^2 + ab + b^2),$$

which you can confirm by integration.

Solution 61

Here is the low-Reynolds-number terminal velocity from the lecture notes:

$$v \sim \frac{2}{9} \frac{gr^2}{\nu} \left(\frac{\rho_{\text{obj}}}{\rho_{\text{fl}}} - 1 \right).$$

Here ρ_{obj} is the density of water, which is much greater than ρ_{fl} , the density of air. So the -1 is not important. With that simplification and calling $2/9 = 1/4$,

$$v \sim \frac{1}{4} \times \frac{10 \text{ m s}^{-2} \times 10^{-10} \text{ m}^2}{10^{-5} \text{ m}^2 \text{ s}^{-1}} \times 1000 \sim 2 \text{ cm s}^{-1}.$$

Solution 62

The Reynolds number is roughly

$$\text{Re} \sim \frac{10^{-5} \text{ m} \times 2 \cdot 10^{-2} \text{ m s}^{-1}}{10^{-5} \text{ m}^2 \text{ s}^{-1}} \sim 0.02.$$

Solution 63

At 2 cm s^{-1} , it takes $5 \cdot 10^4 \text{ s}$ to fall 1 km. A day is roughly 10^5 s , so the fall time is about one-half of a day. The everyday consequence is that fog

settles overnight: You go to sleep with a pea-soup fog, and by the time you wake up, it's mostly settled onto the ground and maybe evaporated as the morning sun warms the ground.

Solution 64

Take $G_2 = r/l$ as a start. I'll put Q in G_1 by trying to construct another volume per time. The simplest is rv . So

$$G_1 \equiv \frac{Q}{rv}.$$

The third group has to contain Δp , otherwise it has nowhere to live, and therefore also has to contain ρ , the only other quantity that has mass among its dimensions. The dimensions of $\Delta p/\rho$ are L^2T^{-2} . The only way to get rid of the T^{-2} but without using Q is to divide by v^2 . The dimensions of $\Delta p/\rho v^2$ are L^{-2} . So a dimensionless group is

$$G_3 \equiv \Delta p \frac{r^2}{\rho v^2}.$$

The general form is

$$\frac{Q}{rv} = f\left(\frac{r}{l}, \Delta p \frac{r^2}{\rho v^2}\right).$$

Solution 65

In this oozing-flow limit, the drag is directly from viscosity. And viscous drag is proportional to velocity. Doubling the pressure difference should double the viscous drag that can be overcome, which should double the speed, which should double the flow rate. So $Q \propto \Delta p$.

Solution 66

Doubling Δp and doubling l is just like connecting two tubes in sequence, each with the original Δp and l . The flow rate from one tube smoothly feeds the second tube. So the flow rate of the combined system, which has the doubled Δp and doubled l , is the same as the flow rate of the original system.

So if $\Delta p/l$ does not change, neither should Q , which means that $Q \propto \Delta p/l$.

Solution 67

Let's guess that $f(G_2, G_3)$ is $G_2^n G_3^m$. To get $Q \propto \Delta p$, the exponent m must be 1. To get $Q \propto \Delta p/l$, the exponent n must also be 1. So

$$\frac{Q}{rv} \sim \frac{r}{l} \frac{\Delta p r^2}{\rho v^2},$$

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or

$$Q \sim \frac{\Delta p}{l} \frac{r^4}{\rho v}.$$

Solution 68

From the preceding parts, the dimensionless constant is

$$C = Q \frac{l}{\Delta p} \frac{\rho v}{r^4}.$$

I used a 27-gauge needle, which has $r \sim 0.1$ mm and $l \sim 3$ cm, attached to a 3cc syringe. I filled the syringe with 1.5cc of water, and placed it upside down on a kitchen scale so that the plunger was touching the scale and the needle was in the air. Then I pushed it downward so that the scale read 200 g and timed how long it took to squeeze out 1cc. It took roughly 30 s.

A 200 g scale reading means a force of $mg \sim 2$ N. It is distributed over the syringe barrel, which has diameter 1 cm, so the pressure is

$$\Delta p \sim \frac{2 \text{ N}}{\frac{\pi}{4} \times 10^{-4} \text{ m}^2} \sim 2.5 \cdot 10^4 \text{ Pa}.$$

Putting in all the numbers:

$$C \sim \frac{10^{-6} \text{ m}^3}{30 \text{ s}} \times \frac{3 \cdot 10^{-2} \text{ m}}{2.5 \cdot 10^4 \text{ Pa}} \times \frac{10^{-3} \text{ kg m}^{-1} \text{ s}^{-1}}{10^{-16} \text{ m}^4} \sim 0.4.$$

The true value is $\pi/8 \approx 0.39$!

Solution 69

There are only two independent dimensions, for example M and LT^{-2} . So four variables result in two groups.

Solution 70

For one group, use the same one as in lecture when we found the acceleration:

$$G_2 \equiv \frac{m_1 - m_2}{m_1 + m_2}.$$

The other group has to contain the tension T. It has dimensions of force, as does $m_1 g$. So we could use $T/m_1 g$. But that choice is not symmetric under interchange of m_1 and m_2 . The other choice $T/m_2 g$ has the same problem. But

$$G_1 = \frac{T}{(m_1 + m_2)g}$$

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is symmetric.

So,

$$\frac{T}{(m_1 + m_2)g} = f\left(\frac{m_1 - m_2}{m_1 + m_2}\right).$$

Solution 71

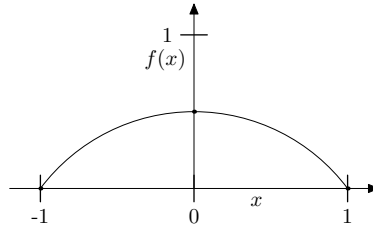
The special cases that helped in guessing the acceleration are $m_1 = 0$, $m_2 = 0$, and $m_1 = m_2$. So use them again.

When one mass is zero, the other mass free falls, so $T = 0$, which means $f(-1) = f(1) = 0$. When $m_1 = m_2$, the system does not accelerate, so the force from the string tension balances the weight, so $T = m_1 g$ and $T = m_2 g$. Therefore $f(0) = 1/2$.

A simple function that passes through these points is

$$f(x) = \frac{1}{2}(1 - x^2).$$

Here is a sketch.



The tension is therefore

$$T = \frac{1}{2}(m_1 + m_2)g \left(1 - \left(\frac{m_1 - m_2}{m_1 + m_2}\right)^2\right) = \frac{2m_1 m_2}{m_1 + m_2}g.$$

This form has an intuitive and compact representation:

$$T = 2(m_1 \parallel m_2)g,$$

where $a \parallel b$ means the parallel combination of a and b as if they were resistances.

Solution 72

The net downward on m_1 is $m_1 g - T$, so its downward acceleration is $g - T/m_1$. The net upward force on m_2 is $T - m_2 g$, so its upward acceleration is $T/m_2 - g$.

These accelerations are equal, so

$$g - \frac{T}{m_1} = \frac{T}{m_2} - g.$$

Collect all the g 's to the left side and all the terms with T on the right side. Then

$$2g = T \left(\frac{1}{m_2} + \frac{1}{m_1} \right).$$

Therefore

$$T = \frac{2m_1 m_2}{m_1 + m_2} g,$$

which matches the result from dimensional analysis and special-cases reasoning.

Solution 73

One method is to use Poiseuille flow by choosing the pressure gradient and pipe diameter to get a slow-enough flow.

One of my plants needs a cup of water ($\sim 200 \text{ cm}^3$) every day, so

$$Q \sim \frac{2 \cdot 10^{-4} \text{ m}^3}{10^5 \text{ s}} \sim 2 \cdot 10^{-9} \text{ m}^3 \text{ s}^{-1}.$$

From an earlier problem,

$$Q \sim \frac{\Delta p}{l} \frac{r^4}{\rho \nu}.$$

To make Q very tiny, the best way is to use a small pipe radius r , because r shows up with a fourth power. I'll see how well $r = 0.1 \text{ mm}$ works.

Another part of the problem is how to make the pressure gradient. I'll let gravity generate the gradient by keeping the water in a tall tank of height h (with a plastic sheet as a cover to prevent evaporation) and using the hydrostatic pressure ρgh as the driving pressure. Then $\Delta p = \rho gh$ and

$$Q \sim \frac{gh}{l} \frac{r^4}{\nu} = g \frac{r^4}{\nu} \frac{h}{l}.$$

With $r = 0.1 \text{ mm}$, the flow rate is

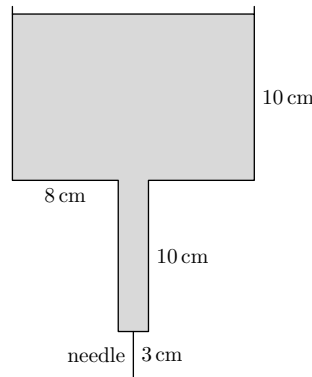
$$Q \sim 10 \text{ m s}^{-2} \times \frac{10^{-16} \text{ m}^4}{10^{-6} \text{ m}^2 \text{ s}^{-1}} \times \frac{h}{l} = 10^{-9} \text{ m}^3 \text{ s}^{-1} \times \frac{h}{l}.$$

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So $h/l \sim 2$ in order to get the desired $Q \sim 2 \cdot 10^{-9} \text{ m}^3 \text{ s}^{-1}$. Actually, if I include the factor of $\pi/8$, then I need $h/l \sim 5$. One way to get $r \sim 0.1 \text{ mm}$ is to use a 27-gauge needle. A typical 27-gauge needle – at least, the ones I’ve used for giving myself allergy treatments – has $l \sim 3 \text{ cm}$. So I’ll need $h \sim 15 \text{ cm}$.

It’s not easy to keep h fixed at 15 cm for two weeks. But if h does not vary too much, then the flow will be constant enough. I’ll let h vary between 10 cm and 20 cm with this arrangement:



The 17 cm width for the big part of the tank allows the tank to contain enough water – roughly 3 liters – to water the plant for a couple weeks.

Solution 74

Resistance shows up in the relation between power and current: $P = I^2 R$. Since the dimensions of I are $Q T^{-1}$, the dimensions of R are

$$[R] = \frac{[P]}{[I^2]} = \frac{M L^2 T^{-3}}{Q^2 T^{-2}} = Q^{-2} M L^2 T^{-1}.$$

Yuk.

Capacitance shows up in $Q = CV$, so

$$[C] = \frac{Q}{[V]}.$$

Voltage is energy per charge (which is why electron-Volts are a unit of energy). So

$$[C] = Q^2 M^{-1} L^{-2} T^2.$$

Inductance shows up in $V = L dI/dt$, so

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$$[L] = \frac{[V]}{[I/t]} = \frac{Q^{-1}ML^2T^{-2}}{QT^{-2}} = Q^{-2}ML^2.$$

Solution 75

I can construct the dimensions of L, R, and C using $Q^{-2}ML^2$ and T.

Solution 76

Reasonable dimensionless group are $\sqrt{L/C}/R$ or its square L/RC^2 . If I use the first choice, it is also known as the quality factor Q (nothing to do with charge), which measures how resonant a circuit is.

Bibliography

- [1] Harold Abelson and Gerald Jay Sussman with Julie Sussman. *Structure and Interpretation of Computer Programs*. MIT Press, 2nd edition, 1996.
- [2] Carl Bernstein and Bob Woodward. *All the President's Men*. Simon and Schuster, New York, 1974.
- [3] E. Buckingham. On physically similar systems. *Physical Review*, 4(4):345–376, 1914.
- [4] J. W. Backus, R. J. Beeber, S. Best, R. Goldberg and L. M. Haibt et al.. The FORTRAN automatic coding system. In *Proceedings of the Western Joint Computing Conference*, pages 188–198, Los Angeles, California, 1957.
- [5] Allen Cadwallader and David Gagnè. *Analysis of Tonal Music: A Schenkerian Approach*. Oxford University Press, New York, 2007.
- [6] S. Chandrasekhar. *An Introduction to the Study of Stellar Structure*. University of Chicago Press, Chicago, Ill., 1939.
- [7] S. Chandrasekhar. *Hydrodynamic and Hydromagnetic Stability*. Oxford University Press, Oxford, 1961.
- [8] Albert Einstein. *The principle of relativity*. Dover Publications, New York, 1952 [1923].
- [9] Albert Einstein. *Relativity: The special and the general theory*. Three Rivers Press, reprint edition, 1995.
- [10] T. E. Faber. *Fluid Dynamics for Physicists*. Cambridge University Press, Cambridge, England, 1995.
- [11] R. P. Feynman. Space-time approach to quantum electrodynamics. *Physical Review*, 76(6):769–789, 1949.
- [12] Richard Feynman, Robert B. Leighton and Matthew Sands. *The Feynman Lectures on Physics*. Addison-Wesley, Reading, MA, 1963.

- [13] Richard P. Feynman. *QED: The Strange Theory of Light and Matter*. Princeton, 1988.
- [14] Mike Gancarz. *The UNIX Philosophy*. Digital Press, 1995.
- [15] Williams James. *The Principles of Psychology*, volume 2. Henry Holt, New York, 1890.
- [16] Edwin T. Jaynes. *Probability Theory: The Logic of Science*. Cambridge University Press, 2003.
- [17] David Kaiser. *Drawing Theories Apart: The Dispersion of Feynman Diagrams in Postwar Physics*. University of Chicago Press, 2005.
- [18] A. Karatsuba and Yu Ofman. Multiplication of many-digital numbers by automatic computers. *Doklady Akad. Nauk SSSR*, 145:293–294, 1962. Translation in *Physics-Doklady* 7:595–596 (1963)..
- [19] Horace Lamb. *Hydrodynamics*. Dover, New York, 6th edition, 1932.
- [20] Niccolo Machiavelli. *The Art of War.*, 1520. <http://etext.library.adelaide.edu.au/m/machiavelli/>
- [21] R. David Middlebrook. Methods of design-oriented analysis: Low-entropy expressions. In *New Approaches to Undergraduate Engineering Education IV*, 1992. See mit.edu/6.969/www/.
- [22] Hermann Minkowski. Raum und Zeit. *Physikalische Zeitschrift*, 10:104–111, 1909.
- [23] Reviel Netz. *The Shaping of Deduction in Greek Mathematics: A Study in Cognitive History*. Cambridge University Press, 1999.
- [24] John Simpson and Edmund Weiner, editors. *Oxford English Dictionary*. Oxford University Press, 2nd edition, 1989. online edition (2008).
- [25] R. C. Pankhurst. *Dimensional Analysis and Scale Factors*. Chapman and Hall, London, 1964.
- [26] E. M. Purcell. Life at low Reynolds number. *American Journal of Physics*, 45:3–11, 1977.
- [27] Vilayanur S. Ramachandran. The artful brain. 2003 online at www.bbc.co.uk/radio4/reith2003/lecture3.shtml.
- [28] Eric S. Raymond. *The Art of UNIX Programming*. Addison-Wesley, 2003.
- [29] K. A. Ross and D. E. Knuth. A programming and problem solving seminar. Technical Report, Stanford University, Stanford, California, 1989.

- [30] Bertrand Russell. *A History of Western Philosophy, and Its Connection with Political and Social Circumstances from the Earliest Times to the Present Day*. Simon and Schuster, New York, 1945.
- [31] Knut Schmid-Nielsen. *Scaling: Why Animal Size is So Important*. Cambridge University Press, Cambridge, England, 1984.
- [32] A. Schönhage and V. Strassen. Schnelle multiplikation grosser zahlen. *Computing*, 7:281–292, 1971.
- [33] Herbert A. Simon and Jill H. Larkin. Why a diagram is (soemtimes) worth ten thousand words. *Cognitive Science*, 11:65–99, 1987.
- [34] Edwin F. Taylor and John Archibald Wheeler. *Spacetime Physics: Introduction to Special Relativity*. W. H. Freeman, 2nd edition, 1992.
- [35] Edward Tufte. *Beautiful Evidence*. Graphics Press, 2006.
- [36] Steven Vogel. *Life in Moving Fluids: The Physical Biology of Flow*. Princeton University Press, Princeton, NJ, 1994.
- [37] Scott Walter. *The Expanding Worlds of General Relativity*, volume 7 of *Einstein Studies*, chapter Minkowski, Mathematicians, and the Mathematical Theory of Relativity, pages 45–86. Birkhauser, Boston/Basel, 1999.

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